

Two loop QCD vertices at the symmetric point

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Abstract. We compute the triple gluon, quark-gluon and ghost-gluon vertices of QCD at the symmetric subtraction point at two loops in the $\overline{\text{MS}}$ scheme. In addition we renormalize each of the three vertices in their respective momentum subtraction schemes, MOMggg, MOMq and MOMh. The conversion functions of all the wave functions, coupling constant and gauge parameter renormalization constants of each of the schemes relative to $\overline{\text{MS}}$ are determined analytically. These are then used to derive the *three* loop anomalous dimensions of the gluon, quark, Faddeev-Popov ghost and gauge parameter as well as the β -function in an arbitrary linear covariant gauge for each MOM scheme. There is good agreement of the latter with earlier Landau gauge numerical estimates of Chetyrkin and Seidensticker.

1 Introduction.

The structure of the renormalization group functions of Quantum Chromodynamics (QCD) has been established to a high degree of precision for well over a decade now. Originally the one loop discovery of asymptotic freedom due to the negative β -function, [1, 2], was swiftly followed by the two loop computation, [3, 4]. Within a decade the three loop term emerged, [5], but the four loop result, [6], took substantially longer to determine. It was later confirmed in [7]. To appreciate how involved the calculation of [6] was, it required the evaluation of the order of 50000 of Feynman diagrams and an intense amount of symbolic algebraic manipulation using the language FORM, [8]. Also new techniques to evaluate four loop Feynman integrals were developed, [8], as well as the coding of a FORM routine to handle the colour group algebra manipulations automatically, [9]. In addition to the β -function governing the running coupling constant the other renormalization group functions have been available at a variety of loop orders over the same time scale, [10, 11, 12, 13, 14, 15]. In summarizing the status of the basic renormalization group functions of QCD we have concentrated on the status of the modified minimal subtraction, $\overline{\text{MS}}$, scheme. This renormalization scheme is the most commonly used for perturbative QCD computations. It is used primarily because it is a mass independent scheme. Consequently one simplifying feature is that all the Feynman graphs which one needs to evaluate to determine the β -function to four loops in this scheme involve single scale 2-point Feynman integrals. This means, for instance, that techniques such as integration by parts can be systematically used to reduce integrals to either basic master topologies or simple chain type integrals whose evaluation is effectively trivial. Indeed the MINCER algorithm, [16], is an excellent example of the implementation of integration by parts and has been coded in FORM, [8]. Briefly MINCER determines massless 2-point integrals at three loops to the finite part with respect to the regulator. The divergences are written as simple poles in ϵ where $d = 4 - 2\epsilon$ and d is the arbitrary spacetime dimension of dimensional regularization. The latter, which we will use, is the main regularization method for perturbative quantum field theories and, for instance, preserves gauge invariance. Whilst MINCER has been used extensively for many problems other than the basic renormalization group functions of QCD, the FORM version, [17], was used to renormalize the QCD coupling constant at three loops, [12]. The reason for this is that one can nullify an external momentum of one of the legs of a 3-point function, without introducing spurious infrared infinities, and hence apply the MINCER algorithm to obtain the correct three loop $\overline{\text{MS}}$ β -function of [5]. Although this has proved to be a powerful technique for renormalization group functions, for more physical problems the $\overline{\text{MS}}$ scheme has several drawbacks.

One of these is the fact that the $\overline{\text{MS}}$ scheme is not a physical scheme. Specifically, the subtraction at the vertex as noted earlier in the MINCER approach is at a point of exceptional momentum. Whilst this is sufficient to extract the divergences and hence find the renormalization constant, one would have potential infrared ambiguities if one were trying to make a measurement of the finite part of the associated vertex function at this exceptional momentum configuration. This has been known for a long time, [18, 19], but has also been remarked upon again more recently in the specific context of Green's functions with an inserted operator, [20]. Indeed the measurement of vertices is a topic of interest as one can study any of the vertices non-perturbatively using say lattice regularization. Such analyses are necessary for determining the strength of the strong coupling constant accurately for comparison with experiment and examining its behaviour at low energy. Therefore, it seems appropriate to consider vertex momentum configurations which are non-exceptional and hence schemes which are physical rather than the unphysical minimal schemes. A set of such schemes was introduced in [21] and given the general designation of momentum subtraction schemes being denoted by MOM. In essence

they consist of two criteria. The first is that 3-point vertices are considered at a symmetric subtraction point, [21]. In other words the squares of the three external momenta are all set equal to each other. Therefore, there is no nullification of an external momentum and hence no exceptional momenta. The second is that at the subtraction point the scheme is defined in such a way that after renormalization there are no $O(a)$ corrections where $a = g^2/(16\pi^2)$ and g is the gauge coupling constant. Thus the renormalization constants all contain finite parts, [21], in addition to the poles in ϵ which must always be subtracted. In [21] the MOM schemes were analysed comprehensively at one loop. By schemes we mean the three types derived from the respective three 3-point vertices in the canonical linear covariant gauge fixing. These are denoted by MOMggg, MOMh and MOMq due to their origin from the respective triple gluon, ghost-gluon and quark-gluon vertices. Indeed these one loop MOM scheme computations of the renormalization group functions have been the state of the art for a long time for a relatively simple reason. This is because for the vertex renormalization the symmetric subtraction point momentum configuration introduces single scale 3-point Feynman integrals, [21]. At one loop there is only one such basic master integral to evaluate which was given in [21]. However, at two loops there are several basic master integrals which were only evaluated analytically in recent years. Therefore, it is the purpose of this article to extend the work of [21] to two loops. We will achieve this by evaluating all 2-point and 3-point vertices to the finite part at two loops at the symmetric subtraction point analytically. Hence we will determine the gluon, Faddeev-Popov ghost and quark wave function and coupling constant renormalization constants for each of the three MOM schemes in an arbitrary linear covariant gauge. En route we will also provide the structure of each of the three vertices to the finite part in the $\overline{\text{MS}}$ scheme at the symmetric point. This information should prove useful to lattice groups seeking to measure any of the vertex functions in the $\overline{\text{MS}}$ scheme. For instance, in such an exercise the lattice results must match onto the ultraviolet part of the vertex function which is where perturbation theory is valid. Equipped with the two loop renormalization constants we will also determine the conversion functions for each scheme including the relation between the coupling constants. Hence via properties of the renormalization group we will determine each MOM scheme renormalization group function to *three* loops in an arbitrary linear covariant gauge.

Finally, we mention other related work in this area. Prior to this article there were approximate calculations of the three MOM β -functions in the Landau gauge, [22]. This was achieved by approximating the basic master two loop integrals at the symmetric point by an expansion where one of the external momentum is marked to produce an asymmetric Green's function. Then the integrals were expanded in the ratio of this marked momentum to the other independent momentum, [22, 23]. So if a sufficient number of terms is computed in this parameter, where the coefficients are rapidly decreasing in size, then a reasonable approximation can be deduced by truncating at an appropriate order, [22, 23]. Moreover, error estimates can be deduced. In this expansion the 3-point functions reduce to 2-point functions. Hence each term in the series is evaluated using the MINCER algorithm. Indeed we previewed the results of this article in [24], by providing the exact three loop QCD MOM β -functions in the Landau gauge and demonstrated how accurate the results were in comparison to [22]. This was a very close overlap which was impressive given the level of computing technology available to the authors of [22] at that time. Next, it would be remiss if we failed to mention a related three loop MOM β -function computation given in [25]. There the MOM β -function was defined using the invariant charge concept of [26, 27]. As it involves only finite parts of 2-point functions it is independent of any of the vertices of the theory unlike the MOMggg, MOMh or MOMq schemes. Hence, aside from the Riemann zeta series, it does not have any of the special number structures, such as harmonic polylogarithms, which derive from the single scale 3-point master integrals and are evident in our analytic results, [24]. Also in this context a variation on this theme of es-

tablishing MOM scheme definitions based solely on 2-point functions has been developed more recently in [28]. Known as the minimal MOM scheme it circumvents a renormalization condition based on the ghost-gluon vertex 3-point function by imposing the alternative condition that the renormalization constant of the ghost-gluon vertex is the *same* as that of the vertex in the $\overline{\text{MS}}$ scheme itself. One benefit of this is that in a linear covariant gauge a non-perturbative running coupling constant can be defined purely in terms of the gluon and ghost propagator form factors. With this definition the running coupling constant can be measured more accurately in principle using lattice gauge theory techniques. For instance, recent activity on this specific aspect can be found in [29, 30]. Given that one can determine the gluon and ghost 2-point functions to several loop orders in perturbation theory, the four loop minimal MOM β -function has been determined in [28]. Though unlike [25] quark mass effects have only been estimated in [28]. Finally, we note that the measurement of vertices non-perturbatively is not exclusively studied by lattice techniques. For instance, recently the triple gluon vertex was examined using Schwinger-Dyson methods, [31]. Therefore, the structure of the two loop vertices given at the symmetric point here should also prove relevant in analyses using those techniques too. Other two loop studies of 3-point vertices of QCD include the results of [32, 33] where the triple gluon vertex was examined in the zero momentum limit and in the on-shell configuration respectively. The latter also includes the same analysis for the ghost-gluon vertex.

The article is organized as follows. We outline the general formalism and notation we use in section 2 as well as discussing various aspects of the renormalization in each of the three MOM schemes in section 3. The results for the $\overline{\text{MS}}$ and MOM amplitudes of the respective vertex functions as well as the conversion functions and the three loop renormalization group functions are given in each of the three following sections*. Finally, we present conclusions in section 7. An appendix contains the explicit tensors of the bases for each of the three vertices as well as the respective projection matrices.

2 Preliminaries.

We begin by discussing the general features of the computation we perform. The three Green's functions we consider are $\langle A_\mu^a(p) A_\nu^b(q) A_\sigma^c(r) \rangle$, $\langle \psi^i(p) \bar{\psi}^j(q) A_\sigma^c(r) \rangle$ and $\langle c^a(p) \bar{c}^b(q) A_\sigma^c(r) \rangle$ where $r = -p - q$ by momentum conservation. The symmetric subtraction point is defined by the condition

$$p^2 = q^2 = r^2 = -\mu^2 \quad (2.1)$$

where μ is the common mass scale. It will also be used as the mass scale to ensure that the coupling constant remains dimensionless in dimensional regularization in d -dimensions which we use throughout. Therefore our results for the finite parts of the vertex functions will not involve logarithms which can be restored from knowledge of the renormalization group functions. From (2.1) we have

$$pq = \frac{1}{2}\mu^2 \quad (2.2)$$

and we will use p and q as the two independent momenta throughout. Their sum will be taken to flow out through a gluon external leg which will therefore be the reference leg in each of the vertices. Given that each vertex has a colour group tensor associated with it, we factor it off when we consider the symmetric point, which we do exclusively from now on, by defining

$$\left\langle A_\mu^a(p) A_\nu^b(q) A_\sigma^c(-p-q) \right\rangle \Big|_{p^2=q^2=-\mu^2} = f^{abc} \Sigma_{\mu\nu\sigma}^{\text{ggg}}(p, q) \Big|_{p^2=q^2=-\mu^2}$$

*All the results presented in the article, including the full analytic forms for an arbitrary gauge, have been included in an attached electronic data file for each of the three MOM schemes.

$$\begin{aligned}
\left\langle \psi^i(p) \bar{\psi}^j(q) A_\sigma^c(-p-q) \right\rangle \Big|_{p^2=q^2=-\mu^2} &= T_{ij}^c \Sigma_\sigma^{\text{qcg}}(p, q) \Big|_{p^2=q^2=-\mu^2} \\
\left\langle c^a(p) \bar{c}^b(q) A_\sigma^c(-p-q) \right\rangle \Big|_{p^2=q^2=-\mu^2} &= f^{abc} \Sigma_\sigma^{\text{ccg}}(p, q) \Big|_{p^2=q^2=-\mu^2} .
\end{aligned} \tag{2.3}$$

We will use ggg, qgg and ccg in equations to denote the triple gluon, quark-gluon and ghost-gluon vertex functions respectively. Next we decompose the Lorentz amplitudes $\Sigma_{\mu\nu\sigma}^{\text{ggg}}(p, q)|_{p^2=q^2=-\mu^2}$, $\Sigma_\sigma^{\text{qcg}}(p, q)|_{p^2=q^2=-\mu^2}$ and $\Sigma_\sigma^{\text{ccg}}(p, q)|_{p^2=q^2=-\mu^2}$ into the scalar amplitudes. The Lorentz tensor basis for each function is not the same. Nor is the choice of basis we will use for the decomposition unique. The explicit forms of the tensors are given in Appendix A. Though we note that away from the symmetric point, where the equalities of (2.1) are no longer valid, then the basis will involve a larger number of tensors. Therefore, we formally define the scalar amplitudes as

$$\begin{aligned}
\Sigma_{\mu\nu\sigma}^{\text{ggg}}(p, q) \Big|_{p^2=q^2=-\mu^2} &= \sum_{k=1}^{14} \mathcal{P}_{(k)\mu\nu\sigma}^{\text{ggg}}(p, q) \Sigma_{(k)}^{\text{ggg}}(p, q) \\
\Sigma_\sigma^{\text{qcg}}(p, q) \Big|_{p^2=q^2=-\mu^2} &= \sum_{k=1}^6 \mathcal{P}_{(k)\sigma}^{\text{qcg}}(p, q) \Sigma_{(k)}^{\text{qcg}}(p, q) \\
\Sigma_\sigma^{\text{ccg}}(p, q) \Big|_{p^2=q^2=-\mu^2} &= \sum_{k=1}^2 \mathcal{P}_{(k)\sigma}^{\text{ccg}}(p, q) \Sigma_{(k)}^{\text{ccg}}(p, q)
\end{aligned} \tag{2.4}$$

where $\Sigma_{(k)}^i(p, q)$ are the scalar amplitudes at (2.1) and $\mathcal{P}_{(k)}^{i\mu_1\cdots\mu_{n_i}}(p, q)$ are the tensors of the respective bases with i corresponding to one of ggg, qgg or ccg. We use k to label the basis elements and have chosen the labelling in such a way that channel 1 corresponds to the tensor which occurs in the Feynman rule of the corresponding vertex in the QCD Lagrangian. Though given the structure of the triple gluon vertex the first six tensors are part of that vertex.

To determine the values of each of the scalar amplitudes we use the method of projection. In other words an identified amplitude can be isolated by multiplying the Green's function by a specific linear combination of the basis tensors. Therefore, we have

$$\begin{aligned}
f^{abc} \Sigma_{(k)}^{\text{ggg}}(p, q) &= \mathcal{M}_{kl}^{\text{ggg}} \left(\mathcal{P}_{(l)}^{\text{ggg}\mu\nu\sigma}(p, q) \left\langle A_\mu^a(p) A_\nu^b(q) A_\sigma^c(-p-q) \right\rangle \right) \Big|_{p^2=q^2=-\mu^2} \\
T_{ij}^c \Sigma_{(k)}^{\text{qcg}}(p, q) &= \mathcal{M}_{kl}^{\text{qcg}} \left(\mathcal{P}_{(l)}^{\text{qcg}\sigma}(p, q) \left\langle \psi^i(p) \bar{\psi}^j(q) A_\sigma^c(-p-q) \right\rangle \right) \Big|_{p^2=q^2=-\mu^2} \\
f^{abc} \Sigma_{(k)}^{\text{ccg}}(p, q) &= \mathcal{M}_{kl}^{\text{ccg}} \left(\mathcal{P}_{(l)}^{\text{ccg}\sigma}(p, q) \left\langle c^a(p) \bar{c}^b(q) A_\sigma^c(-p-q) \right\rangle \right) \Big|_{p^2=q^2=-\mu^2}
\end{aligned} \tag{2.5}$$

for each vertex where we have included the passive colour factor on the left hand side to complement the one which is implicit on the right side. The free spinor indices in the quark-gluon vertex have been left implicit. The matrix \mathcal{M}_{kl}^i is the projection matrix and the explicit forms for each of the vertices are given in the appendix. It is computed by first finding the matrix \mathcal{N}_{kl}^i for each vertex which is constructed from the basis tensors by Lorentz contraction in d -dimensions using the conditions of the symmetric point, (2.1). In other words

$$\mathcal{N}_{kl}^i = \left(\mathcal{P}_{(k)\mu_1\cdots\mu_{n_i}}^i(p, q) \mathcal{P}_{(l)}^{i\mu_1\cdots\mu_{n_i}}(p, q) \right) \Big|_{p^2=q^2=-\mu^2} . \tag{2.6}$$

This produces a matrix, \mathcal{N}_{kl}^i , whose entries are polynomials in d and \mathcal{M}_{kl}^i corresponds to its inverse. For the quark-gluon vertex the tensor basis necessarily has to be built from γ -matrices in addition to the momentum vectors. As we will be working in dimensional regularization we will use the generalized γ -matrices, $\Gamma_{(n)}^{\mu_1\cdots\mu_n}$, [34, 35, 36]. These form a complete set of matrices which span the spinor space of the associated d -dimensional spacetime. They are defined to be completely antisymmetric in the Lorentz indices and are given by

$$\Gamma_{(n)}^{\mu_1\cdots\mu_n} = \gamma^{[\mu_1} \cdots \gamma^{\mu_n]} \tag{2.7}$$

where an overall factor of $1/n!$ is understood and $\Gamma_{(0)}$ is the unit element. One beneficial property, among other general properties [37, 38], is that the trace over the generalized γ -matrices is isotropic as

$$\text{tr} \left(\Gamma_{(m)}^{\mu_1 \dots \mu_m} \Gamma_{(n)}^{\nu_1 \dots \nu_n} \right) \propto \delta_{mn} I^{\mu_1 \dots \mu_m \nu_1 \dots \nu_n} \quad (2.8)$$

and $I^{\mu_1 \dots \mu_m \nu_1 \dots \nu_n}$ is the unit tensor. For the quark-gluon vertex only the $n = 1$ and 3 generalized matrices arise, as we are working in the chiral limit throughout, which can be seen in the explicit decomposition in appendix A.

We have used several main working tools to complete our analysis which are all computer based as it would be virtually impossible to proceed without automatic Feynman diagram generators as well as symbolic manipulation programmes. For each of the three vertex functions the Feynman graphs are constructed with the QGRAF package, [39]. For the triple gluon vertex there are 8 one loop and 106 two loop diagrams contributing to the Green's function. For the other two vertices the number of graphs is the same with 2 one loop and 33 two loop diagrams. From the QGRAF output the Lorentz and colour indices are appended in the symbolic manipulation language FORM, [8]. Indeed FORM is used as the machinery for the rest of our algebraic computations as it is efficient in handling the huge amounts of algebra required for our analysis. To compute the Feynman graphs to the required finite part in dimensional regularization we use the Laporta algorithm, [40]. Briefly the aim is to write each Feynman diagram in the Green's function in terms of a set of basic scalar master integrals whose expressions to the finite part are known. After applying the projection matrix to the Green's function the resulting scalar Feynman integrals are written in a specific format. Given the structure of the QCD propagators and vertices the scalar products in the numerators are rewritten in terms of the propagator denominators in preparation for using the method of [40]. However, given the symmetric point condition then for all the topologies there will be irreducible numerators where there are no corresponding denominators. Equally there will be propagators raised to a power larger than unity. To reduce this very large number of scalar integrals to the set of masters requires an intense amount of integration by parts. One method which achieves this is the Laporta approach, [40]. This systematically determines all the integration by parts relations, as well as Lorentz identity relations, between all the integrals which are needed. The algorithm then uses a systematic way of reducing integrals classified in various levels to the lower levels or to a basic master in that level. The beauty of the method is that it terminates and can be coded for implementation on a computer. There are several available packages. We have chosen to use REDUZE, [41], which is written in the symbolic manipulation formalism of GINAC, [42], whose working language is C++. One main aspect of the package which we exploit is to construct a database of relations between the integrals and then to lift out those we require for our specific computation. These are simply mapped to FORM format and an integration module included within our overall automatic FORM programme. This sums up all the contributions to each of the Green's functions allowing us to perform the renormalization in any of the schemes of interest. For the latter we follow the algorithm for automatic Feynman diagram computations devised in [12]. This involves performing all the integrals as a function of bare parameters with the renormalized values introduced by a simple rescaling via the respective renormalization constants. Therefore, this means that we only compile the results for the vertex amplitudes once prior to determining the $\overline{\text{MS}}$ or MOM scheme renormalization group functions and amplitudes.

Finally, as the analytic results we derive involve cumbersome expressions even in the Landau gauge as will be evident, we will give numerical values for all our results in an arbitrary gauge. As our computations revolve around the symmetric subtraction point the underlying one and two loop scalar master integrals involve structures not seen in the 3-point momentum configuration where one external momentum is nullified. In the latter case at low loop order one ordinarily only encounters rationals and the Riemann zeta function, $\zeta(z)$. For the symmetric point one

finds the function

$$s_n(z) = \frac{1}{\sqrt{3}} \Im \left[\text{Li}_n \left(\frac{e^{iz}}{\sqrt{3}} \right) \right] \quad (2.9)$$

for various arguments where $\text{Li}_n(z)$ is the polylogarithm function as well as one specific combination of the harmonic polylogarithms which we denote by

$$\Sigma = \mathcal{H}_{31}^{(2)} + \mathcal{H}_{43}^{(2)} . \quad (2.10)$$

The master integrals where these originally arise are summarized in [43] but the explicit evaluation are given in a set of articles, [44, 45, 46, 47]. Given that these functions will arise we record the relevant numerical values that we needed which are

$$\begin{aligned} \zeta(3) &= 1.20205690 \quad , \quad \Sigma = 6.34517334 \quad , \quad \psi' \left(\frac{1}{3} \right) = 10.09559713 \quad , \\ \psi''' \left(\frac{1}{3} \right) &= 488.1838167 \quad , \quad s_2 \left(\frac{\pi}{2} \right) = 0.32225882 \quad , \quad s_2 \left(\frac{\pi}{6} \right) = 0.22459602 \quad , \\ s_3 \left(\frac{\pi}{2} \right) &= 0.32948320 \quad , \quad s_3 \left(\frac{\pi}{6} \right) = 0.19259341 \end{aligned} \quad (2.11)$$

where $\psi(z)$ is the derivative of the logarithm of Euler Γ -function.

3 Renormalization.

We devote this section to general aspects of MOM scheme renormalization. Having described the computer algebraic machinery used to construct the vertex functions we now recall how the MOM schemes are defined where we regard the amplitudes of (2.4) as having been determined to the finite part. The divergences are removed into the coupling constant renormalization constant. However, to two loops this is an iterative procedure which is entwined with the 2-point function renormalization. This is because to extract the coupling constant counterterm from the vertex function one has to pay attention to the wave function renormalization constants of the external fields of the vertex function. Therefore, one first determines the one loop wave function renormalization constants in the MOM scheme of interest which is then fixed in examining one loop vertex function defining that particular MOM scheme. For both the 2-point functions and the specific vertex the MOMi renormalization constant is defined so that at the subtraction point there are no $O(a)$ corrections after the renormalization constant is defined. Throughout we will use the syntax that in MOMi or equations i represents ggg, q or h. Once the one loop renormalization constants are fully determined then one repeats the exercise for the two loop contribution to first the 2-point function and then the associated vertex. The reason for explicitly defining the procedure is to ensure that there is no inconsistency in determining the coupling constant renormalization constant. The finite parts of the one loop wave function renormalization constants impact upon the finite parts of the two loop MOMi coupling constant renormalization constants, [21]. Otherwise an inconsistency in the renormalization group would emerge in trying to deduce the anomalous dimensions as well as the associated conversion functions for each renormalization constant. As a check on the vertex functions we have computed, we have verified that the two loop $\overline{\text{MS}}$ coupling constant renormalization constant of [1, 2, 3, 4] correctly emerges when renormalizing at the symmetric subtraction point. One final point concerning our MOMi scheme renormalizations and that is that we first determine all the amplitudes before setting the renormalization constants for each scheme. For MOMi schemes we render that channel with the ϵ divergences to have no $O(a)$ corrections. However, we emphasise that this is by no means the only way of defining the renormalization constants

within the MOM ethos. An alternative, for instance, is to first multiply the vertex function with its Lorentz tensor structure present, by the tensor of the corresponding Feynman rules. Then the coupling constant renormalization constant is defined by ensuring that there are no $O(a)$ corrections to this object. We note that doing this for each of the three schemes would involve several of the non-Feynman rule amplitudes in contributing to the coupling constant renormalization. We have not proceeded in this way as it does not appear to be in keeping with [21, 22]. However, in providing the full vertex structure in terms of the Lorentz tensors in the $\overline{\text{MS}}$ scheme an interested reader has the opportunity to study such alternative MOM scheme definitions. Indeed it may be the case that convergence of certain perturbative series could be improved in such a way.

Having reviewed the procedure we followed we now comment on the relation of the parameters of the theory in different schemes. For QCD the relevant parameters are the coupling constant and the linear covariant gauge fixing parameter. As the former is vertex dependent we comment on the latter first. As outlined the MOMi renormalizations require 2-point function renormalization. Therefore, like the coupling constant, [21], the gauge parameter can be different in different schemes. To relate them we follow the standard method and define

$$\alpha_{\text{MOMi}}(\mu) = \frac{Z_A^{\text{MOMi}}}{Z_A^{\overline{\text{MS}}}} \alpha_{\overline{\text{MS}}}(\mu) \quad (3.1)$$

where the subscript on the parameter refers to the scheme the variable is defined with respect to and Z_A is the gluon wave function renormalization constant. We follow the same conventions as [48] in defining the gauge parameter renormalization constant, Z_α , by

$$\alpha_o = Z_\alpha^{-1} Z_A \alpha \quad (3.2)$$

where the subscript, o, indicates the bare parameter. With this convention then the respective anomalous dimensions satisfy, [48],

$$\gamma_\alpha(a, \alpha) = -\gamma_A(a, \alpha) \quad (3.3)$$

in each scheme and $\gamma_\alpha(a, \alpha)$ is the anomalous dimension of the linear covariant gauge parameter. In carrying out the renormalization in each of the three schemes we have determined $\alpha_{\text{MOMi}}(\mu)$ for each of the three cases and found that using the MOM scheme definition of [21]

$$\begin{aligned} \alpha_{\text{MOMi}} = & \left[1 + \left[80T_F N_f - 9\alpha_{\overline{\text{MS}}}^2 - 18\alpha_{\overline{\text{MS}}} - 97 \right] C_A \right] \frac{a_{\overline{\text{MS}}}}{36} \\ & + \left[\left[18\alpha_{\overline{\text{MS}}}^4 - 18\alpha_{\overline{\text{MS}}}^3 + 190\alpha_{\overline{\text{MS}}}^2 - 576\zeta(3)\alpha_{\overline{\text{MS}}} + 463\alpha_{\overline{\text{MS}}} + 864\zeta(3) - 7143 \right] C_A^2 \right. \\ & - \left[320\alpha_{\overline{\text{MS}}}^2 + 320\alpha_{\overline{\text{MS}}} - 2304\zeta(3) - 4248 \right] C_A T_F N_f \\ & \left. - [4608\zeta(3) - 5280] C_F T_F N_f \right] \frac{a_{\overline{\text{MS}}}^2}{288} + O\left(\alpha_{\overline{\text{MS}}}^3\right) \alpha_{\overline{\text{MS}}} . \end{aligned} \quad (3.4)$$

To ensure that the mapping is not divergent due to poles in ϵ one has to iteratively solve (3.1) order by order in perturbation theory hand in hand with the coupling constant of that scheme. This is because when we set the renormalization constants in a scheme both parameters of the explicit forms belong to that particular scheme. From (3.4) we see that the gauge parameter mapping is the same for all three schemes. Indeed it is the same as that for the RI' scheme, [48]. This is not unexpected as the parameter mapping is effectively the conversion function and reflects an underlying feature of the renormalization group. In essence it tracks how the scheme is defined for that one parameter amidst the renormalization of all the other parameters

and wave functions within the Green's functions. This will become evident later for other renormalizations. Therefore, given this feature the three loop term of (3.4) has already been given in [48]. Further, we recall that it means that the Landau gauge is preserved between the schemes.

The procedure to define the relation between the coupling constants in different schemes is similar. Though as we are dealing with three MOM schemes then the definitions are different in each case. We follow [21] and [22] for this and define

$$\begin{aligned}
a_{\text{MOMggg}}(\mu) &= a_{\overline{\text{MS}}}(\mu) \left[\frac{\Pi_g^{\text{MOMggg}}(p)}{\Pi_g^{\overline{\text{MS}}}(p)} \right]^3 \bigg|_{p^2 = -\mu^2} \left[\frac{\Sigma_{(1)\overline{\text{MS}}}^{\text{ggg}}(-\mu^2, -\mu^2)}{\Sigma_{(1)\text{MOMggg}}^{\text{ggg}}(-\mu^2, -\mu^2)} \right]^2 \\
a_{\text{MOMq}}(\mu) &= a_{\overline{\text{MS}}}(\mu) \left[\frac{\Pi_g^{\text{MOMq}}(p) \left(\Sigma_q^{\text{MOMq}}(p) \right)^2}{\Pi_g^{\overline{\text{MS}}}(p) \left(\Sigma_q^{\overline{\text{MS}}}(p) \right)^2} \right] \bigg|_{p^2 = -\mu^2} \left[\frac{\Sigma_{(1)\overline{\text{MS}}}^{\text{qqg}}(-\mu^2, -\mu^2)}{\Sigma_{(1)\text{MOMq}}^{\text{qqg}}(-\mu^2, -\mu^2)} \right]^2 \\
a_{\text{MOMh}}(\mu) &= a_{\overline{\text{MS}}}(\mu) \left[\frac{\Pi_g^{\text{MOMh}}(p) \left(\Sigma_c^{\text{MOMh}}(p) \right)^2}{\Pi_g^{\overline{\text{MS}}}(p) \left(\Sigma_c^{\overline{\text{MS}}}(p) \right)^2} \right] \bigg|_{p^2 = -\mu^2} \left[\frac{\Sigma_{(1)\overline{\text{MS}}}^{\text{gcc}}(-\mu^2, -\mu^2)}{\Sigma_{(1)\text{MOMh}}^{\text{gcc}}(-\mu^2, -\mu^2)} \right]^2 \quad (3.5)
\end{aligned}$$

where we use the same designation for the vertices as before. Clearly the definitions involve the respective vertex functions evaluated at the symmetric subtraction point. Moreover, the amplitude chosen is that which has the divergences in ϵ prior to renormalization or equivalently the amplitude which corresponds to the vertex Feynman rule. In the case of the triple gluon vertex we have chosen channel 1 which is only part of the Feynman rule. However, as will be apparent from the explicit results the other amplitudes from 2 to 6 are related in the way one would expect from the vertex structure so that our definition is consistent. Whilst in each of the three definitions the corresponding amplitude in the MOMi schemes have no $O(a)$ corrections, we have formally included it to ensure the normalization is correct and that the ratio of the vertex amplitudes from MOMi to $\overline{\text{MS}}$ begins with unity. The other main feature of (3.5) is the presence of the 2-point functions. Specifically $\Pi_g(p)$, $\Sigma_c(p)$ and $\Sigma_q(p)$ are respectively the scalar amplitudes of the gluon polarization and the Faddeev-Popov ghost and quark self-energies in the various schemes. The particular combination of which of these appears follows from the vertex of that MOMi scheme. In deriving the perturbative relations between these two parameters one has to proceed iteratively order by order in perturbation theory paying attention to the gauge parameter mapping in the same scheme at the same time.

One particular property of the gauge parameter and coupling constant mapping between the MOMi and $\overline{\text{MS}}$ schemes is that we can now construct the other conversion functions for the wave function renormalizations and the coupling constant itself. Whilst the latter is not unrelated to (3.5) we note that we regard conversion functions as being derived from the explicit forms of the renormalization constants themselves in the two schemes to be consistent with other work. Therefore, since we define the coupling constant renormalization constant, Z_g , by,

$$g_o = \mu^\epsilon Z_g g \quad (3.6)$$

then the conversion functions are given by

$$C_g^{\text{MOMi}}(a, \alpha) = \frac{Z_g^{\text{MOMi}}}{Z_g^{\overline{\text{MS}}}} \quad , \quad C_\phi^{\text{MOMi}}(a, \alpha) = \frac{Z_\phi^{\text{MOMi}}}{Z_\phi^{\overline{\text{MS}}}} \quad (3.7)$$

where $\phi \in \{A, \psi, c\}$. We will record the explicit forms of these for each of our schemes but note that to determine them we follow the same iterative procedure as we did in deriving (3.4). One

main benefit of the conversion functions and the parameter mappings is that we can deduce the β -function and anomalous dimensions to *three* loops from the renormalization group. Specifically, [22],

$$\beta^{\text{MOMi}}(a_{\text{MOMi}}, \alpha_{\text{MOMi}}) = \left[\beta^{\overline{\text{MS}}}(a_{\overline{\text{MS}}}) \frac{\partial a_{\text{MOMi}}}{\partial a_{\overline{\text{MS}}}} + \alpha_{\overline{\text{MS}}} \gamma_{\alpha}^{\overline{\text{MS}}}(a_{\overline{\text{MS}}}, \alpha_{\overline{\text{MS}}}) \frac{\partial a_{\text{MOMi}}}{\partial \alpha_{\overline{\text{MS}}}} \right]_{\overline{\text{MS}} \rightarrow \text{MOMi}} \quad (3.8)$$

and

$$\begin{aligned} \gamma_{\phi}^{\text{MOMi}}(a_{\text{MOMi}}, \alpha_{\text{MOMi}}) = & \left[\gamma_{\phi}^{\overline{\text{MS}}}(a_{\overline{\text{MS}}}) + \beta^{\overline{\text{MS}}}(a_{\overline{\text{MS}}}) \frac{\partial}{\partial a_{\overline{\text{MS}}}} \ln C_{\phi}^{\text{MOMi}}(a_{\overline{\text{MS}}}, \alpha_{\overline{\text{MS}}}) \right. \\ & \left. + \alpha_{\overline{\text{MS}}} \gamma_{\alpha}^{\overline{\text{MS}}}(a_{\overline{\text{MS}}}, \alpha_{\overline{\text{MS}}}) \frac{\partial}{\partial \alpha_{\overline{\text{MS}}}} \ln C_{\phi}^{\text{MOMi}}(a_{\overline{\text{MS}}}, \alpha_{\overline{\text{MS}}}) \right]_{\overline{\text{MS}} \rightarrow \text{MOMi}} \end{aligned} \quad (3.9)$$

where the MOMi β -functions will depend on the gauge parameter and only be scheme independent at one loop as these are mass dependent renormalization schemes, [21]. Though the $\overline{\text{MS}}$ β -function is independent of α which is why it has only one argument. The mapping $\overline{\text{MS}} \rightarrow \text{MOMi}$ indicates that the object within the square brackets is first computed in terms of $\overline{\text{MS}}$ variables and then these variables are mapped back to the MOMi scheme variables by inverting (3.4) and those derived from (3.5). As a check on this procedure we have calculated the β -function and anomalous dimensions to two loops for each MOM scheme directly from the two loop renormalization constants and verified that the computation contained on the right hand sides of (3.8) and (3.9) are in agreement at the same order. This only requires the one loop terms of the conversion functions and therefore this provides a check on our computer routines designed to do this automatically. Finally, in order to ease comparison of our three loop MOMi results with the $\overline{\text{MS}}$ scheme in the same notation the numerical expressions for $SU(3)$ for the latter scheme are, [1, 2, 3, 4, 5],

$$\begin{aligned} \beta^{\overline{\text{MS}}}(a) &= - [11.0000000 - 0.6666667N_f]a^2 \\ &\quad - [102.0000000 - 12.6666667N_f]a^3 \\ &\quad - [1428.5000000 - 279.6111111N_f + 6.0185185N_f^2]a^4 + O(a^5) \\ \gamma_A^{\overline{\text{MS}}}(a, \alpha) &= [0.6666667N_f - 6.5000000 + 1.5000000\alpha]a \\ &\quad - [66.3750000 - 12.3750000\alpha - 2.2500000\alpha^2 - 10.1666667N_f]a^2 \\ &\quad - [915.9625108 - 165.2479023\alpha - 33.9291631\alpha^2 - 5.9062500\alpha^3 \\ &\quad - [186.8599000 - 9.0000000\alpha]N_f + 7.9629630N_f^2]a^3 + O(a^4) \\ \gamma_c^{\overline{\text{MS}}}(a, \alpha) &= [0.7500000\alpha - 2.2500000]a \\ &\quad - [17.8125000 + 0.5625000\alpha - 1.2500000N_f]a^2 \\ &\quad - [256.2687446 - 2.1729239\alpha + 0.5114565\alpha^2 - 1.2656250\alpha^3 \\ &\quad - [46.2756056 - 3.9375000\alpha]N_f - 0.9722222N_f^2]a^3 + O(a^4) \\ \gamma_{\psi}^{\overline{\text{MS}}}(a, \alpha) &= 1.3333333\alpha a \\ &\quad + [22.3333333 + 8.0000000\alpha + 1.0000000\alpha^2 - 1.3333333N_f]a^2 \\ &\quad + [528.3243079 + 109.4435121\alpha + 20.0342561\alpha^2 + 3.7500000\alpha^3 \\ &\quad - [61.1111111 + 8.5000000\alpha]N_f + 0.7407407N_f^2]a^3 + O(a^4) \end{aligned} \quad (3.10)$$

to the same numerical accuracy. To ease the presentation on the eye we will use the convention that when the scheme appears on the function on the left hand side then the variables, such as the coupling constant and gauge parameter, are the variables in the *same* scheme. The exception to this is the conversion functions, $C_g^{\text{MOMi}}(a, \alpha)$ and $C_\phi^{\text{MOMi}}(a, \alpha)$, where the arguments are the $\overline{\text{MS}}$ variables.

4 Triple gluon vertex.

We now begin the mundane task of recording our results for each scheme and its respective vertices by concentrating on the triple gluon vertex first. As indicated earlier the full analytic versions of all the results for an arbitrary gauge, in this and the next two sections, have been included in a separate data file. The $\overline{\text{MS}}$ $SU(3)$ numerical values for the amplitudes are

$$\begin{aligned}
\Sigma_{(1)}^{\text{ggg}}(p, q)|_{\overline{\text{MS}}} &= \Sigma_{(2)}^{\text{ggg}}(p, q)|_{\overline{\text{MS}}} = -\frac{1}{2} \Sigma_{(3)}^{\text{ggg}}(p, q)|_{\overline{\text{MS}}} = -\Sigma_{(4)}^{\text{ggg}}(p, q)|_{\overline{\text{MS}}} \\
&= \frac{1}{2} \Sigma_{(5)}^{\text{ggg}}(p, q)|_{\overline{\text{MS}}} = -\Sigma_{(6)}^{\text{ggg}}(p, q)|_{\overline{\text{MS}}} \\
&= -1 - \left[1.1212444 - 3.7618956\alpha - 1.2890232\alpha^2 + 0.1250000\alpha^3 \right. \\
&\quad \left. - 0.0417366N_f \right] a \\
&\quad + \left[29.7530676 + 16.4600770\alpha - 9.7794300\alpha^2 - 3.2060809\alpha^3 \right. \\
&\quad \left. - 1.6522848\alpha^4 + 0.2812500\alpha^5 \right. \\
&\quad \left. - [11.5677203 - 0.9686976\alpha - 0.9112399\alpha^2 + 0.4166667\alpha^3]N_f \right] a^2 \\
&\quad + O(a^3) \\
\Sigma_{(7)}^{\text{ggg}}(p, q)|_{\overline{\text{MS}}} &= 2 \Sigma_{(9)}^{\text{ggg}}(p, q)|_{\overline{\text{MS}}} = -2 \Sigma_{(11)}^{\text{ggg}}(p, q)|_{\overline{\text{MS}}} = -\Sigma_{(14)}^{\text{ggg}}(p, q)|_{\overline{\text{MS}}} \\
&= \left[7.0567163 - 3.3280464\alpha - 0.5079304\alpha^2 + 0.0573179\alpha^3 - 1.0926858N_f \right] a \\
&\quad + \left[116.0789643 - 13.6830818\alpha + 0.3484134\alpha^2 + 4.7763124\alpha^3 \right. \\
&\quad \left. + 0.8908609\alpha^4 - 0.1289652\alpha^5 \right. \\
&\quad \left. - [20.2710109 + 1.0153018\alpha - 0.5745217\alpha^2 - 0.1910596\alpha^3]N_f \right] a^2 \\
&\quad + O(a^3) \\
\Sigma_{(8)}^{\text{ggg}}(p, q)|_{\overline{\text{MS}}} &= -\Sigma_{(13)}^{\text{ggg}}(p, q)|_{\overline{\text{MS}}} \\
&= \left[7.3683002 - 3.3518377\alpha - 0.5701159\alpha^2 + 0.1926821\alpha^3 - 1.2130096N_f \right] a \\
&\quad + \left[126.0048710 - 11.8048854\alpha + 3.7795690\alpha^2 + 4.3779190\alpha^3 \right. \\
&\quad \left. + 1.2887087\alpha^4 - 0.4335348\alpha^5 \right. \\
&\quad \left. - [23.5898191 - 0.0155813\alpha - 0.9363317\alpha^2 - 0.6422738\alpha^3]N_f \right] a^2 \\
&\quad + O(a^3) \\
\Sigma_{(10)}^{\text{ggg}}(p, q)|_{\overline{\text{MS}}} &= -\Sigma_{(12)}^{\text{ggg}}(p, q)|_{\overline{\text{MS}}} \\
&= -\left[0.3115839 - 0.0237913\alpha - 0.0621855\alpha^2 + 0.1353643\alpha^3 - 0.1203238N_f \right] a \\
&\quad - \left[9.9259067 + 1.8781964\alpha + 3.4311557\alpha^2 - 0.3983934\alpha^3 \right. \\
&\quad \left. + 0.3978478\alpha^4 - 0.3045696\alpha^5 \right. \\
&\quad \left. - [3.3188082 - 1.0308831\alpha - 0.3618100\alpha^2 - 0.4512142\alpha^3]N_f \right] a^2
\end{aligned}$$

$$+ O(a^3) . \quad (4.11)$$

We have indicated the relations between amplitudes of the various projection tensor channels. These are consistent with the expectations for the structure of the vertex from symmetry given that we have evaluated the vertex function at the symmetric point. In addition in this context

$$\Sigma_{(7)}^{ggg}(p, q) \Big|_{\overline{\text{MS}}} = \Sigma_{(8)}^{ggg}(p, q) \Big|_{\overline{\text{MS}}} - \Sigma_{(10)}^{ggg}(p, q) \Big|_{\overline{\text{MS}}} \quad (4.12)$$

which we checked was true to two loops prior to the numerical evaluation. In Table 1 we give the comparison of our results for the channel 1 amplitudes to those of [22]. In this and other comparisons we have used the coupling constant convention of the presentation in [22] where the series was in powers of $\alpha_s = g^2/(4\pi^2)$. This is to retain the error estimates given in [22]. It is clear that our evaluation is comfortably close to the approximations of [22].

$\Sigma_{(1),2}^{ggg}(p, q) \Big _{\overline{\text{MS}}}$	C_A^2	$C_A T_F N_f$	$C_F T_F N_f$
Ref [22]	0.22(4)	- 0.65(7)	0.408(10)
This paper	0.2066185	- 0.6620808	0.4052081

Table 1. Comparison of channel 1 two loop Landau gauge amplitude with [22] by colour factor.

The MOMggg scheme amplitudes satisfy the same relations and in particular we have

$$\begin{aligned}
\Sigma_{(1)}^{ggg}(p, q) \Big|_{\text{MOMggg}} &= \Sigma_{(2)}^{ggg}(p, q) \Big|_{\text{MOMggg}} = - \frac{1}{2} \Sigma_{(3)}^{ggg}(p, q) \Big|_{\text{MOMggg}} \\
&= - \Sigma_{(4)}^{ggg}(p, q) \Big|_{\text{MOMggg}} = \frac{1}{2} \Sigma_{(5)}^{ggg}(p, q) \Big|_{\text{MOMggg}} \\
&= - \Sigma_{(6)}^{ggg}(p, q) \Big|_{\text{MOMggg}} = - 1 + O(a^3) \\
\Sigma_{(7)}^{ggg}(p, q) \Big|_{\text{MOMggg}} &= 2 \Sigma_{(9)}^{ggg}(p, q) \Big|_{\text{MOMggg}} = - 2 \Sigma_{(11)}^{ggg}(p, q) \Big|_{\text{MOMggg}} \\
&= - \Sigma_{(14)}^{ggg}(p, q) \Big|_{\text{MOMggg}} \\
&= \left[7.0567163 - 3.3280464\alpha - 0.5079304\alpha^2 + 0.0573179\alpha^3 \right. \\
&\quad \left. - 1.0926858N_f \right] a \\
&\quad - \left[78.7833165 - 99.1996625\alpha + 10.0012225\alpha^2 + 10.9109237\alpha^3 \right. \\
&\quad \left. - 1.2024953\alpha^4 - 0.2831609\alpha^5 + 0.0214942\alpha^6 \right. \\
&\quad \left. - [34.3080792 - 16.2422875\alpha - 1.8203921\alpha^2 + 0.6079935\alpha^3]N_f \right. \\
&\quad \left. + 3.7791007N_f^2 \right] a^2 + O(a^3) \\
\Sigma_{(8)}^{ggg}(p, q) \Big|_{\text{MOMggg}} &= - \Sigma_{(13)}^{ggg}(p, q) \Big|_{\text{MOMggg}} \\
&= \left[7.3683002 - 3.3518377\alpha - 0.5701159\alpha^2 + 0.1926821\alpha^3 \right. \\
&\quad \left. - 1.2130096N_f \right] a \\
&\quad - \left[77.4614044 - 103.6568232\alpha + 5.5514994\alpha^2 + 12.5463340\alpha^3 \right. \\
&\quad \left. - 2.9431069\alpha^4 - 0.5253739\alpha^5 + 0.07225580\alpha^6 \right. \\
&\quad \left. - [35.3894861 - 16.0837324\alpha - 1.7300353\alpha^2 + 1.1212780\alpha^3]N_f \right. \\
&\quad \left. + 4.1952457N_f^2 \right] a^2 + O(a^3)
\end{aligned}$$

$$\begin{aligned}
\Sigma_{(10)}^{\text{ggg}}(p, q) \Big|_{\text{MOMggg}} &= - \Sigma_{(12)}^{\text{ggg}}(p, q) \Big|_{\text{MOMggg}} \\
&= - \left[0.3115839 - 0.0237913\alpha - 0.0621855\alpha^2 + 0.1353643\alpha^3 \right. \\
&\quad \left. - 0.1203238N_f \right] a \\
&\quad - \left[1.3219120 + 4.4571607\alpha + 4.44972304\alpha^2 - 1.6354103\alpha^3 \right. \\
&\quad \left. + 1.7406116\alpha^4 + 0.2422130\alpha^5 - 0.0507616\alpha^6 \right. \\
&\quad \left. + [1.0814069 + 0.1585516\alpha + 0.0903568\alpha^2 + 0.5132845\alpha^3]N_f \right. \\
&\quad \left. - 0.4161497N_f^2 \right] a^2 + O(a^3)
\end{aligned} \tag{4.13}$$

with the corresponding relation

$$\Sigma_{(7)}^{\text{ggg}}(p, q) \Big|_{\text{MOMggg}} = \Sigma_{(8)}^{\text{ggg}}(p, q) \Big|_{\text{MOMggg}} - \Sigma_{(10)}^{\text{ggg}}(p, q) \Big|_{\text{MOMggg}} \tag{4.14}$$

also being true to two loops analytically. Given the nature of the MOMggg scheme the relations for the amplitudes of channels 1 to 6 demonstrate that our renormalization is consistent and that our projection has been implemented consistently within our FORM programmes as well as being a check on our REDUZE database. Next we record the four conversion functions for the wave function and coupling constant renormalizations in the MOMggg scheme relative to the $\overline{\text{MS}}$ scheme. These are

$$\begin{aligned}
C_g^{\text{MOMggg}}(a, \alpha) &= 1 - \left[13.2462444 - 1.5118956\alpha - 0.1640232\alpha^2 \right. \\
&\quad \left. + 0.1250000\alpha^3 - 1.7084032N_f \right] a \\
&\quad - \left[217.0368707 + 36.7247782\alpha + 7.0535877\alpha^2 - 1.1557619\alpha^3 \right. \\
&\quad \left. + 1.0453915\alpha^4 - 0.0996192\alpha^5 - 0.0156250\alpha^6 \right. \\
&\quad \left. - [33.1527255 + 3.7086335\alpha - 0.0047429\alpha^2 - 0.6354341\alpha^3]N_f \right. \\
&\quad \left. - 0.5342658N_f^2 \right] a^2 + O(a^3) \\
C_A^{\text{MOMggg}}(a, \alpha) &= 1 + \left[8.0833333 + 1.5000000\alpha + 0.7500000\alpha^2 - 1.1111111N_f \right] a \\
&\quad + \left[256.1034914 + 31.4182743\alpha + 8.4375000\alpha^2 + 2.8125000\alpha^3 \right] \\
&\quad - [53.9129277 + 1.6666667\alpha]N_f + 1.2345679N_f^2 \Big] a^2 + O(a^3) \\
C_c^{\text{MOMggg}}(a, \alpha) &= 1 + 3.0000000a \\
&\quad + \left[80.9357699 + 3.0725670\alpha + 1.3465290\alpha^2 - 5.9375000N_f \right] a^2 \\
&\quad + O(a^3) \\
C_\psi^{\text{MOMggg}}(a, \alpha) &= 1 - 1.3333333\alpha a \\
&\quad - \left[25.4642061 + 11.5753172\alpha + 2.7222222\alpha^2 - 2.3333333N_f \right] a^2 \\
&\quad + O(a^3)
\end{aligned} \tag{4.15}$$

where we note that the expressions for all bar $C_g^{\text{MOMggg}}(a, \alpha)$ are formally the same as those for the RI' scheme. By definition $C_g^{\text{RI}'}(a, \alpha)$ is unity. From $C_g^{\text{MOMggg}}(a, \alpha)$ we can deduce the explicit form of the coupling constant mapping. Though as the full form is cumbersome we record the analytic version for the Landau gauge which is

$$a_{\text{MOMggg}} = a_{\overline{\text{MS}}} + \left[\left[69\psi'(\frac{1}{3}) - 46\pi^2 + 1188 \right] C_A + \left[128\pi^2 - 192\psi'(\frac{1}{3}) - 432 \right] T_F N_f \right] \frac{a_{\overline{\text{MS}}}^2}{162}$$

$$\begin{aligned}
& + \left[\left[19044(\psi'(\tfrac{1}{3}))^2 - 25392\pi^2\psi'(\tfrac{1}{3}) - 6938784\psi'(\tfrac{1}{3}) - 100602\psi'''(\tfrac{1}{3}) \right. \right. \\
& \quad - 72643392s_2(\tfrac{\pi}{6}) + 145286784s_2(\tfrac{\pi}{2}) + 121072320s_3(\tfrac{\pi}{6}) - 96857856s_3(\tfrac{\pi}{2}) \\
& \quad + 276736\pi^4 + 4625856\pi^2 - 113724\Sigma + 8301852\zeta(3) + 40126833 \\
& \quad \left. \left. - 504468\frac{\ln^2(3)\pi}{\sqrt{3}} + 6053616\frac{\ln(3)\pi}{\sqrt{3}} + 541836\frac{\pi^3}{\sqrt{3}} \right] C_A^2 \right. \\
& + \left[141312\pi^2\psi'(\tfrac{1}{3}) - 105984(\psi'(\tfrac{1}{3}))^2 - 2960064\psi'(\tfrac{1}{3}) + 33592320s_2(\tfrac{\pi}{6}) \right. \\
& \quad - 67184640s_2(\tfrac{\pi}{2}) - 55987200s_3(\tfrac{\pi}{6}) + 44789760s_3(\tfrac{\pi}{2}) - 47104\pi^4 \\
& \quad + 1973376\pi^2 + 2239488\Sigma - 8957952\zeta(3) - 26695008 \\
& \quad \left. \left. + 233280\frac{\ln^2(3)\pi}{\sqrt{3}} - 2799360\frac{\ln(3)\pi}{\sqrt{3}} - 250560\frac{\pi^3}{\sqrt{3}} \right] C_A T_F N_f \right. \\
& + \left[124416\psi'''(\tfrac{1}{3}) - 1492992\psi'(\tfrac{1}{3}) - 331776\pi^4 + 995328\pi^2 \right. \\
& \quad - 4478976\Sigma + 6718464\zeta(3) - 7138368] C_F T_F N_f \\
& + \left[147456(\psi'(\tfrac{1}{3}))^2 - 196608\pi^2\psi'(\tfrac{1}{3}) + 2322432\psi'(\tfrac{1}{3}) + 65536\pi^4 \right. \\
& \quad \left. \left. - 1548288\pi^2 + 2923776 \right] T_F^2 N_f^2 \right] \frac{a_{\overline{\text{MS}}}^3}{419904} + O\left(a_{\overline{\text{MS}}}^4\right). \tag{4.16}
\end{aligned}$$

By contrast the full gauge dependent version in numerical form for $SU(3)$ is

$$\begin{aligned}
a_{\text{MOMggg}} &= a_{\overline{\text{MS}}} \\
& + \left[26.4924889 - 3.0237913\alpha_{\overline{\text{MS}}} - 0.3280464\alpha_{\overline{\text{MS}}}^2 \right. \\
& \quad \left. + 0.2500000\alpha_{\overline{\text{MS}}}^3 - 3.4168064N_f \right] a_{\overline{\text{MS}}}^2 \\
& + \left[960.4627167 - 46.7120794\alpha_{\overline{\text{MS}}} + 7.9285132\alpha_{\overline{\text{MS}}}^2 + 9.1110752\alpha_{\overline{\text{MS}}}^3 \right. \\
& \quad + 1.0375721\alpha_{\overline{\text{MS}}}^4 - 0.3222558\alpha_{\overline{\text{MS}}}^5 + 0.0156250\alpha_{\overline{\text{MS}}}^6 \\
& \quad \left. - \left[202.0850109 - 8.0802973\alpha_{\overline{\text{MS}}} - 1.6907923\alpha_{\overline{\text{MS}}}^2 \right. \right. \\
& \quad \left. \left. + 0.0104341\alpha_{\overline{\text{MS}}}^3 \right] N_f + 7.6873930N_f^2 \right] a_{\overline{\text{MS}}}^3 + O\left(a_{\overline{\text{MS}}}^4\right). \tag{4.17}
\end{aligned}$$

Equipped with these we have computed the three loop renormalization group functions for the MOMggg scheme. Again given space considerations we record the Landau gauge expressions for each case. They are

$$\begin{aligned}
\beta^{\text{MOMggg}}(a, 0) &= - \left[\frac{11}{3}C_A - \frac{4}{3}T_F N_f \right] a^2 - \left[\frac{34}{3}C_A^2 - 4C_F T_F N_f - \frac{20}{3}C_A T_F N_f \right] a^3 \\
& + \left[\left[209484(\psi'(\tfrac{1}{3}))^2 - 279312\pi^2\psi'(\tfrac{1}{3}) + 37087200\psi'(\tfrac{1}{3}) \right. \right. \\
& \quad + 368874\psi'''(\tfrac{1}{3}) + 266359104s_2(\tfrac{\pi}{6}) - 532718208s_2(\tfrac{\pi}{2}) \\
& \quad - 443931840s_3(\tfrac{\pi}{6}) + 355145472s_3(\tfrac{\pi}{2}) - 890560\pi^4 \\
& \quad - 24724800\pi^2 + 416988\Sigma - 30440124\zeta(3) - 51650217 \\
& \quad \left. \left. + 1849716\frac{\ln^2(3)\pi}{\sqrt{3}} - 22196592\frac{\ln(3)\pi}{\sqrt{3}} - 1986732\frac{\pi^3}{\sqrt{3}} \right] C_A^3 \right. \\
& + \left[1656000\pi^2\psi'(\tfrac{1}{3}) - 1242000(\psi'(\tfrac{1}{3}))^2 - 38988864\psi'(\tfrac{1}{3}) \right. \\
& \quad - 134136\psi'''(\tfrac{1}{3}) - 220029696s_2(\tfrac{\pi}{6}) + 440059392s_2(\tfrac{\pi}{2}) \\
& \quad \left. \left. + 366716160s_3(\tfrac{\pi}{6}) - 293372928s_3(\tfrac{\pi}{2}) - 194304\pi^4 \right] \right.
\end{aligned}$$

$$\begin{aligned}
& + 25992576\pi^2 - 8363088\Sigma + 43914960\zeta(3) + 49845132 \\
& - 1527984\frac{\ln^2(3)\pi}{\sqrt{3}} + 18335808\frac{\ln(3)\pi}{\sqrt{3}} + 1641168\frac{\pi^3}{\sqrt{3}} \Big] C_A^2 T_F N_f \\
& + \Big[2045952(\psi'(\tfrac{1}{3}))^2 - 2727936\pi^2 \psi'(\tfrac{1}{3}) + 11591424\psi'(\tfrac{1}{3}) \\
& + 44789760s_2(\tfrac{\pi}{6}) - 89579520s_2(\tfrac{\pi}{2}) - 74649600s_3(\tfrac{\pi}{6}) \\
& + 59719680s_3(\tfrac{\pi}{2}) + 909312\pi^4 - 7727616\pi^2 \\
& + 2985984\Sigma - 11943936\zeta(3) - 8460288 \\
& + 311040\frac{\ln^2(3)\pi}{\sqrt{3}} - 3732480\frac{\ln(3)\pi}{\sqrt{3}} - 334080\frac{\pi^3}{\sqrt{3}} \Big] C_A T_F^2 N_f^2 \\
& + \Big[786432\pi^2 \psi'(\tfrac{1}{3}) - 589824(\psi'(\tfrac{1}{3}))^2 - 442368\psi'(\tfrac{1}{3}) \\
& - 262144\pi^4 + 294912\pi^2 - 82944 \Big] T_F^3 N_f^3 \\
& + \Big[4758912\psi'(\tfrac{1}{3}) - 456192\psi'''(\tfrac{1}{3}) + 1216512\pi^4 - 3172608\pi^2 \\
& + 16422912\Sigma - 24634368\zeta(3) + 23421312 \Big] C_A C_F T_F N_f \\
& + \Big[165888\psi'''(\tfrac{1}{3}) - 442368\pi^4 - 5971968\Sigma + 8957952\zeta(3) \\
& - 7091712 \Big] C_F T_F^2 N_f^2 - 839808 C_F^2 T_F N_f \Big] \frac{a^4}{419904} \\
& + O(a^5)
\end{aligned} \tag{4.18}$$

$$\begin{aligned}
\gamma_A^{\text{MOMggg}}(a, 0) = & - [13C_A - 8T_F N_f] \frac{a}{6} \\
& + \Big[[1794\psi'(\tfrac{1}{3}) - 1196\pi^2 - 2655] C_A^2 + 7776C_F T_F N_f \\
& + [4064\pi^2 - 6096\psi'(\tfrac{1}{3}) + 2304] C_A T_F N_f \\
& + [3072\psi'(\tfrac{1}{3}) - 2048\pi^2 + 1152] T_F^2 N_f^2 \Big] \frac{a^2}{1944} \\
& + \Big[[2310672\pi^2 \psi'(\tfrac{1}{3}) - 1733004(\psi'(\tfrac{1}{3}))^2 - 121373424\psi'(\tfrac{1}{3}) \\
& - 1307826\psi'''(\tfrac{1}{3}) - 944364096s_2(\tfrac{\pi}{6}) + 1888728192s_2(\tfrac{\pi}{2}) \\
& + 1573940160s_3(\tfrac{\pi}{6}) - 1259152128s_3(\tfrac{\pi}{2}) + 2717312\pi^4 \\
& + 80915616\pi^2 - 1478412\Sigma + 164768580\zeta(3) - 117299583 \\
& - 6558084\frac{\ln^2(3)\pi}{\sqrt{3}} + 78697008\frac{\ln(3)\pi}{\sqrt{3}} + 7043868\frac{\pi^3}{\sqrt{3}} \Big] C_A^3 \\
& + [1071108(\psi'(\tfrac{1}{3}))^2 - 14281344\pi^2 \psi'(\tfrac{1}{3}) + 134602560\psi'(\tfrac{1}{3}) \\
& + 804816\psi'''(\tfrac{1}{3}) + 1017847296s_2(\tfrac{\pi}{6}) - 2035694592s_2(\tfrac{\pi}{2}) \\
& - 1696412160s_3(\tfrac{\pi}{6}) + 1357129728s_3(\tfrac{\pi}{2}) + 2614272\pi^4 \\
& - 89735040\pi^2 + 30023136\Sigma - 100567008\zeta(3) + 80188056 \\
& + 7068384\frac{\ln^2(3)\pi}{\sqrt{3}} - 84820608\frac{\ln(3)\pi}{\sqrt{3}} - 7591968\frac{\pi^3}{\sqrt{3}} \Big] C_A^2 T_F N_f \\
& + [25804800\pi^2 \psi'(\tfrac{1}{3}) - 19353600(\psi'(\tfrac{1}{3}))^2 - 40849920\psi'(\tfrac{1}{3}) \\
& - 268738560s_2(\tfrac{\pi}{6}) + 537477120s_2(\tfrac{\pi}{2}) \\
& + 447897600s_3(\tfrac{\pi}{6}) - 358318080s_3(\tfrac{\pi}{2}) - 8601600\pi^4 \\
& + 27233280\pi^2 - 17915904\Sigma + 17915904\zeta(3) + 2052864
\end{aligned}$$

$$\begin{aligned}
& - 1866240 \frac{\ln^2(3)\pi}{\sqrt{3}} + 22394880 \frac{\ln(3)\pi}{\sqrt{3}} + 2004480 \frac{\pi^3}{\sqrt{3}} \Big] C_A T_F^2 N_f^2 \\
& + \left[8257536 (\psi'(\frac{1}{3}))^2 - 11010048 \pi^2 \psi'(\frac{1}{3}) + 6193152 \psi'(\frac{1}{3}) + 3670016 \pi^4 \right. \\
& \quad \left. - 4128768 \pi^2 + 1161216 \right] T_F^3 N_f^3 \\
& + \left[1617408 \psi'''(\frac{1}{3}) - 27993600 \psi'(\frac{1}{3}) - 4313088 \pi^4 + 18662400 \pi^2 \right. \\
& \quad \left. - 58226688 \Sigma - 147806208 \zeta(3) + 125971200 \right] C_A C_F T_F N_f \\
& + \left[35831808 \psi'(\frac{1}{3}) - 995328 \psi'''(\frac{1}{3}) + 2654208 \pi^4 - 23887872 \pi^2 \right. \\
& \quad \left. + 35831808 \Sigma + 53747712 \zeta(3) - 47029248 \right] C_F T_F^2 N_f^2 \\
& - 5038848 C_F^2 T_F N_f \Big] \frac{a^3}{2519424} + O(a^4) \tag{4.19}
\end{aligned}$$

$$\begin{aligned}
\gamma_c^{\text{MOMggg}}(a, 0) &= -\frac{3}{4} C_A a \\
&+ \left[\left[138 \psi'(\frac{1}{3}) - 92 \pi^2 - 63 \right] C_A \right. \\
&\quad \left. + \left[256 \pi^2 - 384 \psi'(\frac{1}{3}) + 72 \right] T_F N_f \right] \frac{C_A a^2}{432} \\
&+ \left[\left[177744 \pi^2 \psi'(\frac{1}{3}) - 133308 (\psi'(\frac{1}{3}))^2 - 9492336 \psi'(\frac{1}{3}) \right. \right. \\
&\quad - 100602 \psi'''(\frac{1}{3}) - 72643392 s_2(\frac{\pi}{6}) + 145286784 s_2(\frac{\pi}{2}) \\
&\quad + 121072320 s_3(\frac{\pi}{6}) - 96857856 s_3(\frac{\pi}{2}) + 209024 \pi^4 \\
&\quad + 6328224 \pi^2 - 113724 \Sigma + 11993508 \zeta(3) - 9641835 \\
&\quad \left. - 504468 \frac{\ln^2(3)\pi}{\sqrt{3}} + 6053616 \frac{\ln(3)\pi}{\sqrt{3}} + 541836 \frac{\pi^3}{\sqrt{3}} \right] C_A^2 \\
&\quad + \left[741888 (\psi'(\frac{1}{3}))^2 - 989184 \pi^2 \psi'(\frac{1}{3}) + 5019840 \psi'(\frac{1}{3}) \right. \\
&\quad + 33592320 s_2(\frac{\pi}{6}) - 67184640 s_2(\frac{\pi}{2}) \\
&\quad - 55987200 s_3(\frac{\pi}{6}) + 44789760 s_3(\frac{\pi}{2}) + 329728 \pi^4 \\
&\quad - 3346560 \pi^2 + 2239488 \Sigma - 5318784 \zeta(3) + 6350400 \\
&\quad \left. + 233280 \frac{\ln^2(3)\pi}{\sqrt{3}} - 2799360 \frac{\ln(3)\pi}{\sqrt{3}} - 250560 \frac{\pi^3}{\sqrt{3}} \right] C_A T_F N_f \\
&\quad + \left[1376256 \pi^2 \psi'(\frac{1}{3}) - 1032192 (\psi'(\frac{1}{3}))^2 - 110592 \psi'(\frac{1}{3}) \right. \\
&\quad \left. - 458752 \pi^4 + 73728 \pi^2 - 1762560 \right] T_F^2 N_f^2 \\
&\quad + \left[124416 \psi'(\frac{1}{3}) - 1492992 \psi'(\frac{1}{3}) - 331776 \pi^4 \right. \\
&\quad \left. + 995328 \pi^2 - 4478976 \Sigma + 1399680 \right] C_F T_F N_f \Big] \frac{C_A a^3}{559872} \\
&+ O(a^4) \tag{4.20}
\end{aligned}$$

and

$$\begin{aligned}
\gamma_\psi^{\text{MOMggg}}(a, 0) &= [25 C_A - 6 C_F - 8 T_F N_f] \frac{C_F a^2}{4} \\
&+ \left[4600 \pi^2 - 6900 \psi'(\frac{1}{3}) - 39690 \zeta(3) + 61011 \right] C_A^2 \\
&+ \left[1656 \psi'(\frac{1}{3}) - 1104 \pi^2 + 15552 \zeta(3) - 23760 \right] C_A C_F
\end{aligned}$$

$$\begin{aligned}
& + \left[21408\psi'(\tfrac{1}{3}) - 14272\pi^2 + 10368\zeta(3) - 28800 \right] C_A T_F N_f \\
& + \left[3072\pi^2 - 4608\psi'(\tfrac{1}{3}) - 4320 \right] C_F T_F N_f + 1944 C_F^2 \\
& + \left[4096\pi^2 - 6144\psi'(\tfrac{1}{3}) + 1152 \right] T_F^2 N_f^2 \left] \frac{C_F a^2}{1296} \right. \\
& \left. + O(a^4) \right. .
\end{aligned} \tag{4.21}$$

That for the quark anomalous dimension is relatively compact as there is no one loop term in the Landau gauge. The explicit gauge dependence is given in the numerical evaluations for $SU(3)$ which are

$$\begin{aligned}
\beta^{\text{MOMggg}}(a, \alpha) &= - [11.0000000 - 0.6666667 N_f] a^2 \\
&- \left[102.0000000 + 19.6546434\alpha - 0.2710840\alpha^2 - 5.8591391\alpha^3 \right. \\
&\quad \left. + 1.1250000\alpha^4 \right. \\
&\quad \left. - \left[12.6666667 + 2.0158609\alpha + 0.4373952\alpha^2 - 0.5000000\alpha^3 \right] N_f \right] a^3 \\
&- \left[1570.9843804 + 658.0709292\alpha + 269.2238338\alpha^2 + 43.0029610\alpha^3 \right. \\
&\quad \left. - 99.2797189\alpha^4 + 14.8550247\alpha^5 + 5.3345924\alpha^6 - 0.7031250\alpha^7 \right. \\
&\quad \left. + \left[0.5659290 - 43.2393672\alpha - 22.7471960\alpha^2 - 19.8709555\alpha^3 \right. \right. \\
&\quad \left. \left. + 14.8347569\alpha^4 + 0.9764184\alpha^5 - 0.2812500\alpha^6 \right] N_f \right. \\
&\quad \left. - \left[67.0895364 + 4.6479610\alpha + 0.8898051\alpha^2 - 2.3056953\alpha^3 \right] N_f^2 \right. \\
&\quad \left. + 2.6581155 N_f^3 \right] a^4 + O(a^5) \\
\gamma_A^{\text{MOMggg}}(a, \alpha) &= [0.6666667 N_f - 6.5000000 + 1.5000000\alpha] a \\
&+ \left[16.9095110 - 41.6433767\alpha + 6.1533855\alpha^2 + 0.9920696\alpha^3 \right. \\
&\quad \left. - 0.3750000\alpha^4 \right. \\
&\quad \left. - \left[12.0931233 - 5.4744039\alpha + 0.2813024\alpha^2 + 0.1666667\alpha^3 \right] N_f \right. \\
&\quad \left. + 1.5371302 N_f^2 \right] a^2 \\
&- \left[1308.9386744 - 647.9260677\alpha + 376.2301295\alpha^2 + 6.3971133\alpha^3 \right. \\
&\quad \left. - 33.0162468\alpha^4 + 7.3253130\alpha^5 + 1.0008734\alpha^6 - 0.1640625\alpha^7 \right. \\
&\quad \left. - \left[491.4309500 - 302.3530495\alpha + 52.3029150\alpha^2 + 6.3604344\alpha^3 \right. \right. \\
&\quad \left. \left. - 6.7153148\alpha^4 - 0.1288604\alpha^5 + 0.0729167\alpha^6 \right] N_f \right. \\
&\quad \left. + \left[74.9190172 - 29.3999306\alpha + 1.4158902\alpha^2 + 1.3449889\alpha^3 \right] N_f^2 \right. \\
&\quad \left. - 6.2022694 N_f^3 \right] a^3 + O(a^4) \\
\gamma_c^{\text{MOMggg}}(a, \alpha) &= [0.7500000\alpha - 2.2500000] a \\
&+ \left[8.7955100 - 21.1728971\alpha + 2.6547391\alpha^2 + 1.3710348\alpha^3 \right. \\
&\quad \left. - 0.1875000\alpha^4 - \left[4.4378145 - 1.7292715\alpha \right] N_f \right] a^2 \\
&- \left[548.8492387 - 436.6720556\alpha + 199.2938032\alpha^2 + 32.7086138\alpha^3 \right. \\
&\quad \left. - 30.0123943\alpha^4 + 1.4303427\alpha^5 + 0.9535617\alpha^6 - 0.0820312\alpha^7 \right. \\
&\quad \left. - \left[157.4669179 - 127.5457558\alpha + 20.0522192\alpha^2 + 7.7592757\alpha^3 \right. \right.
\end{aligned}$$

$$\begin{aligned}
& - 1.5131126\alpha^4] N_f + [19.9741163 - 6.9775531\alpha] N_f^2] a^3 + O(a^4) \\
\gamma_\psi^{\text{MOMggg}}(a, \alpha) = & 1.3333333\alpha a \\
& + [22.3333333 - 10.5455407\alpha + 9.0317217\alpha^2 + 1.4373952\alpha^3 \\
& - 0.3333333\alpha^4 - [1.3333333 - 3.0742604\alpha] N_f] a^2 \\
& - [94.7943290 - 204.1998798\alpha + 218.8404110\alpha^2 - 30.4216658\alpha^3 \\
& - 34.4073860\alpha^4 + 6.3159939\alpha^5 + 1.2577208\alpha^6 - 0.1458333\alpha^7 \\
& - [76.8672720 - 80.5601965\alpha + 53.0718118\alpha^2 + 7.0576676\alpha^3 \\
& - 2.6899779\alpha^4] N_f + [5.2596320 - 12.4045389\alpha] N_f^2] a^3 \\
& + O(a^4) .
\end{aligned} \tag{4.22}$$

We recall, [22], that unlike the $\overline{\text{MS}}$ scheme the three loop term of the MOMggg β -function is cubic in N_f and not quadratic.

5 Ghost-gluon vertex.

We repeat this exercise now for the structure of the ghost-gluon vertex and the associated MOMh renormalization scheme. Though there are fewer amplitudes than for the triple gluon vertex. For the $\overline{\text{MS}}$ scheme the two amplitudes are

$$\begin{aligned}
\Sigma_{(1)}^{\text{cg}}(p, q)|_{\overline{\text{MS}}} = & - 1 - [2.2324710 + 0.3280464\alpha - 0.1464920\alpha^2] a \\
& - [49.2999213 + 16.6398011\alpha + 2.3538026\alpha^2 + 0.0885284\alpha^3 \\
& + 0.1098707\alpha^4 - [4.0701546 + 0.4453256\alpha + 0.1627713\alpha^2] N_f] a^2 \\
& + O(a^3) \\
\Sigma_{(2)}^{\text{cg}}(p, q)|_{\overline{\text{MS}}} = & [1.4824710 - 0.1640232\alpha - 0.1464920\alpha^2] a \\
& + [35.1253580 + 2.2852097\alpha - 0.3277110\alpha^2 + 0.0885284\alpha^3 \\
& + 0.1098707\alpha^4 - [3.2368212 + 0.4453256\alpha + 0.1627713\alpha^2] N_f] a^2 \\
& + O(a^3)
\end{aligned} \tag{5.23}$$

leading to the MOMh scheme expressions

$$\begin{aligned}
\Sigma_{(1)}^{\text{cg}}(p, q)|_{\text{MOMh}} = & - 1 + O(a^3) \\
\Sigma_{(2)}^{\text{cg}}(p, q)|_{\text{MOMh}} = & [1.4824710 - 0.1640232\alpha - 0.1464920\alpha^2] a \\
& + [4.3185039 + 0.6852156\alpha + 0.0493136\alpha^2 - 0.0591276\alpha^3 \\
& - 0.0643817\alpha^4 - [1.5896312 + 0.4453256\alpha] N_f] a^2 \\
& + O(a^3) .
\end{aligned} \tag{5.24}$$

As channel 1 contained the poles in ϵ after MOMh renormalization then in that scheme there are no corrections at the symmetric subtraction point. Unlike the other two vertices we do not include a table comparing our numerical expressions with those of [22] as there is no direct

comparison[†]. Moreover, we have examined whether various combinations of our amplitudes can produce agreement, because of potentially different conventions of defining the basis, but have failed to find any. For other quantities related to the MOMh scheme such as the coupling constant mapping and some coefficients of the three loop $SU(3)$ Landau gauge β -function we find similar but minor discrepancies with [22] which suggest a consistent typographical error in the presentation of certain equations in [22]. Further, related inconsistencies for the results for this vertex given in [22] have also been noted in [28]. Whilst we will comment in more depth in section 7 in the context of the results of the other two schemes we note that our benchmark check, [24], on the Landau gauge $SU(3)$ β -function for MOMh was slightly *better* than that of the MOMgg case.

Continuing with the presentation of our results, the $SU(3)$ numerical values for the respective MOMh conversion functions to $\overline{\text{MS}}$ are

$$\begin{aligned}
C_g^{\text{MOMh}}(a, \alpha) &= 1 - \left[9.2741377 + 1.0780464\alpha + 0.2285058\alpha^2 - 0.5555556N_f \right] a \\
&\quad - \left[191.9555891 + 18.6287007\alpha + 1.8982822\alpha^2 + 0.7339954\alpha^3 \right. \\
&\quad \left. - 0.0675921\alpha^4 - [27.3210789 - 0.1535891\alpha - 0.3808430\alpha^2]N_f \right. \\
&\quad \left. + 0.1543210N_f^2 \right] a^2 + O(a^3) \\
C_A^{\text{MOMh}}(a, \alpha) &= 1 + \left[8.0833333 + 1.5000000\alpha + 0.7500000\alpha^2 - 1.1111111N_f \right] a \\
&\quad + \left[256.1034914 + 31.4182743\alpha + 8.4375000\alpha^2 + 2.8125000\alpha^3 \right] \\
&\quad - [53.9129277 - 1.6666667\alpha]N_f + 1.2345679N_f^2 \Big] a^2 + O(a^3) \\
C_c^{\text{MOMh}}(a, \alpha) &= 1 + 3.0000000a \\
&\quad + \left[80.9357699 + 3.0725670\alpha + 1.3465290\alpha^2 - 5.9375000N_f \right] a^2 \\
&\quad + O(a^3) \\
C_\psi^{\text{MOMh}}(a, \alpha) &= 1 - 1.3333333\alpha a \\
&\quad - \left[25.4642061 + 11.5753172\alpha + 2.7222222\alpha^2 - 2.3333333N_f \right] a^2 \\
&\quad + O(a^3) .
\end{aligned} \tag{5.25}$$

The analytic form of the first of these leads to the coupling constant mapping

$$\begin{aligned}
a_{\text{MOMh}} &= a_{\overline{\text{MS}}} + \left[\left[15\psi'(\tfrac{1}{3}) - 10\pi^2 + 615 \right] C_A - 240T_F N_f \right] \frac{a_{\overline{\text{MS}}}^2}{108} \\
&\quad + \left[\left[450(\psi'(\tfrac{1}{3}))^2 - 600\pi^2\psi'(\tfrac{1}{3}) - 458928\psi'(\tfrac{1}{3}) - 3213\psi'''(\tfrac{1}{3}) \right. \right. \\
&\quad \left. - 3825792s_2(\tfrac{\pi}{6}) + 7651584s_2(\tfrac{\pi}{2}) + 6376320s_3(\tfrac{\pi}{6}) - 5101056s_3(\tfrac{\pi}{2}) \right. \\
&\quad \left. + 8768\pi^4 + 305952\pi^2 + 7776\Sigma + 153576\zeta(3) + 6521760 \right. \\
&\quad \left. - 26568\frac{\ln^2(3)\pi}{\sqrt{3}} + 318816\frac{\ln(3)\pi}{\sqrt{3}} + 28536\frac{\pi^3}{\sqrt{3}} \right] C_A^2 + 460800T_F^2 N_f^2 \\
&\quad + \left[206784\psi'(\tfrac{1}{3}) + 1492992s_2(\tfrac{\pi}{6}) - 2985984s_2(\tfrac{\pi}{2}) - 2488320s_3(\tfrac{\pi}{6}) \right. \\
&\quad \left. + 1990656s_3(\tfrac{\pi}{2}) - 137856\pi^2 - 995328\zeta(3) - 4015296 \right. \\
&\quad \left. + 10368\frac{\ln^2(3)\pi}{\sqrt{3}} - 124416\frac{\ln(3)\pi}{\sqrt{3}} - 11136\frac{\pi^3}{\sqrt{3}} \right] C_A T_F N_f \\
&\quad + [1492992\zeta(3) - 1710720] C_F T_F N_f \Big] \frac{a_{\overline{\text{MS}}}^3}{93312} + O\left(\frac{a_{\overline{\text{MS}}}^4}{93312}\right)
\end{aligned} \tag{5.26}$$

[†]We are grateful to Prof. Chetyrkin for comments on this point.

or numerically, for an arbitrary linear covariant gauge,

$$\begin{aligned}
a_{\text{MOMh}} = & a_{\overline{\text{MS}}} + \left[18.5482754 + 2.1560928\alpha_{\overline{\text{MS}}} + 0.4570116\alpha_{\overline{\text{MS}}}^2 - 1.1111111N_f \right] a_{\overline{\text{MS}}}^2 \\
& + \left[641.9400674 + 97.2451047\alpha_{\overline{\text{MS}}} + 19.9982818\alpha_{\overline{\text{MS}}}^2 + 2.9460299\alpha_{\overline{\text{MS}}}^3 \right. \\
& \quad \left. + 0.0214606\alpha_{\overline{\text{MS}}}^4 - \left[85.5559502 + 3.2863098\alpha_{\overline{\text{MS}}} \right] N_f \right. \\
& \quad \left. + 1.2345679N_f^2 \right] a_{\overline{\text{MS}}}^3 + O\left(a_{\overline{\text{MS}}}^4\right). \tag{5.27}
\end{aligned}$$

Using the renormalization group the three loop renormalization group functions emerge. Analytically in the Landau gauge we have

$$\begin{aligned}
\beta^{\text{MOMh}}(a, 0) = & - \left[\frac{11}{3}C_A - \frac{4}{3}T_F N_f \right] a^2 - \left[\frac{34}{3}C_A^2 - 4C_F T_F N_f - \frac{20}{3}C_A T_F N_f \right] a^3 \\
& + \left[4950(\psi'(\tfrac{1}{3}))^2 - 6600\pi^2\psi'(\tfrac{1}{3}) + 2370816\psi'(\tfrac{1}{3}) \right. \\
& \quad + 11781\psi'''(\tfrac{1}{3}) + 14027904s_2(\tfrac{\pi}{6}) - 28055808s_2(\tfrac{\pi}{2}) - 23379840s_3(\tfrac{\pi}{6}) \\
& \quad + 18703872s_3(\tfrac{\pi}{2}) - 29216\pi^4 - 1580544\pi^2 - 28512\Sigma - 563112\zeta(3) \\
& \quad - 11733336 + 97416\frac{\ln^2(3)\pi}{\sqrt{3}} - 1168992\frac{\ln(3)\pi}{\sqrt{3}} - 104632\frac{\pi^3}{\sqrt{3}} \left. \right] C_A^3 \\
& + \left[2400\pi^2\psi'(\tfrac{1}{3}) - 1800(\psi'(\tfrac{1}{3}))^2 - 1864512\psi'(\tfrac{1}{3}) - 4284\psi'''(\tfrac{1}{3}) \right. \\
& \quad - 10575360s_2(\tfrac{\pi}{6}) + 21150720s_2(\tfrac{\pi}{2}) + 17625600s_3(\tfrac{\pi}{6}) \\
& \quad - 14100480s_3(\tfrac{\pi}{2}) + 10624\pi^4 + 1243008\pi^2 + 10368\Sigma + 3854304\zeta(3) \\
& \quad + 9722592 - 73440\frac{\ln^2(3)\pi}{\sqrt{3}} + 881280\frac{\ln(3)\pi}{\sqrt{3}} + 78880\frac{\pi^3}{\sqrt{3}} \left. \right] C_A^2 T_F N_f \\
& + \left[352512\psi'(\tfrac{1}{3}) + 1990656s_2(\tfrac{\pi}{6}) - 3981312s_2(\tfrac{\pi}{2}) - 3317760s_3(\tfrac{\pi}{6}) \right. \\
& \quad + 2654208s_3(\tfrac{\pi}{2}) - 235008\pi^2 - 1327104\zeta(3) - 1368576 \\
& \quad + 13824\frac{\ln^2(3)\pi}{\sqrt{3}} - 165888\frac{\ln(3)\pi}{\sqrt{3}} - 14848\frac{\pi^3}{\sqrt{3}} \left. \right] C_A T_F^2 N_f^2 \\
& + \left[34560\pi^2 - 51840\psi'(\tfrac{1}{3}) - 5474304\zeta(3) + 6272640 \right] C_A C_F T_F N_f \\
& + \left[1990656\zeta(3) - 1907712 \right] C_F T_F^2 N_f^2 - 186624C_F^2 T_F N_f \left. \right] \frac{a^4}{93312} \\
& + O(a^5) \tag{5.28}
\end{aligned}$$

$$\begin{aligned}
\gamma_A^{\text{MOMh}}(a, 0) = & - \left[13C_A - 8T_F N_f \right] \frac{a}{6} \\
& + \left[\left[195\psi'(\tfrac{1}{3}) - 130\pi^2 - 3186 \right] C_A^2 + 2592C_F T_F N_f \right. \\
& \quad \left. + \left[80\pi^2 - 120\psi'(\tfrac{1}{3}) + 2808 \right] C_A T_F N_f \right] \frac{a^2}{648} \\
& + \left[\left[54600\pi^2\psi'(\tfrac{1}{3}) - 40950(\psi'(\tfrac{1}{3}))^2 - 7120224\psi'(\tfrac{1}{3}) - 41769\psi'''(\tfrac{1}{3}) \right. \right. \\
& \quad - 49735296s_2(\tfrac{\pi}{6}) + 99470592s_2(\tfrac{\pi}{2}) + 82892160s_3(\tfrac{\pi}{6}) - 66313728s_3(\tfrac{\pi}{2}) \\
& \quad + 93184\pi^4 + 4746816\pi^2 + 101088\Sigma + 14628600\zeta(3) - 37070136 \\
& \quad - 345384\frac{\ln^2(3)\pi}{\sqrt{3}} + 1414608\frac{\ln(3)\pi}{\sqrt{3}} + 370968\frac{\pi^3}{\sqrt{3}} \left. \right] C_A^3 \\
& \quad \left. + \left[25200(\psi'(\tfrac{1}{3}))^2 - 33600\pi^2\psi'(\tfrac{1}{3}) + 7615296\psi'(\tfrac{1}{3}) + 25704\psi'''(\tfrac{1}{3}) \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + 50015232s_2(\frac{\pi}{6}) - 100030464s_2(\frac{\pi}{2}) - 83358720s_3(\frac{\pi}{6}) \\
& + 66686976s_3(\frac{\pi}{2}) - 57344\pi^4 - 5076864\pi^2 - 62208\Sigma \\
& + 4121280\zeta(3) + 35848360 + 347328\frac{\ln^2(3)\pi}{\sqrt{3}} \\
& - 4167936\frac{\ln(3)\pi}{\sqrt{3}} - 373056\frac{\pi^3}{\sqrt{3}} \Big] C_A^2 T_F N_f - 1119744 C_F^2 T_F N_f \\
& + [23887872s_2(\frac{\pi}{2}) - 2115072\psi'(\frac{1}{3}) - 11943936s_2(\frac{\pi}{6}) + 19906560s_3(\frac{\pi}{6}) \\
& - 15925248s_3(\frac{\pi}{2}) + 1410048\pi^2 - 3981312\zeta(3) - 5225472 \\
& - 82944\frac{\ln^2(3)\pi}{\sqrt{3}} + 995328\frac{\ln(3)\pi}{\sqrt{3}} + 89088\frac{\pi^3}{\sqrt{3}}] C_A T_F^2 N_f^2 \\
& + [414720\pi^2 - 622080\psi'(\frac{1}{3}) - 32845824\zeta(3) + 33716736] C_A C_F T_F N_f \\
& + [11943936\zeta(3) - 11446272] C_F T_F^2 N_f^2 \Big] \frac{a^3}{559872} + O(a^4) \quad (5.29)
\end{aligned}$$

$$\begin{aligned}
\gamma_c^{\text{MOMh}}(a, 0) = & -\frac{3}{4}C_A a + \left[[15\psi'(\frac{1}{3}) - 10\pi^2 - 198] C_A + 72T_F N_f \right] \frac{C_A a^2}{144} \\
& + \left[[4200\pi^2\psi'(\frac{1}{3}) - 3150(\psi'(\frac{1}{3}))^2 - 559008\psi'(\frac{1}{3}) - 3213\psi'''(\frac{1}{3}) \right. \\
& - 3825792s_2(\frac{\pi}{6}) + 7651584s_2(\frac{\pi}{2}) + 6376320s_3(\frac{\pi}{6}) - 5101056s_3(\frac{\pi}{2}) \\
& + 7168\pi^4 + 372672\pi^2 + 7776\Sigma + 973944\zeta(3) - 2855736 \\
& - 26568\frac{\ln^2(3)\pi}{\sqrt{3}} + 318816\frac{\ln(3)\pi}{\sqrt{3}} + 28536\frac{\pi^3}{\sqrt{3}}] C_A^2 \\
& + [247104\psi'(\frac{1}{3}) + 1492992s_2(\frac{\pi}{6}) - 2985984s_2(\frac{\pi}{2}) - 2488320s_3(\frac{\pi}{6}) \\
& + 1990656s_3(\frac{\pi}{2}) - 164736\pi^2 - 186624\zeta(3) + 2260224 \\
& + 10368\frac{\ln^2(3)\pi}{\sqrt{3}} - 124416\frac{\ln(3)\pi}{\sqrt{3}} - 11136\frac{\pi^3}{\sqrt{3}}] C_A T_F N_f \\
& \left. - 414720T_F^2 N_f^2 + 186624C_F T_F N_f \right] \frac{C_A a^3}{124416} + O(a^4) \quad (5.30)
\end{aligned}$$

and

$$\begin{aligned}
\gamma_\psi^{\text{MOMh}}(a, 0) = & [25C_A - 6C_F - 8T_F N_f] \frac{C_F a^2}{4} \\
& + \left[[500\pi^2 - 750\psi'(\frac{1}{3}) - 13230\zeta(3) + 29187] C_A^2 \right. \\
& + [180\psi'(\frac{1}{3}) - 120\pi^2 + 5184\zeta(3) - 10044] C_A C_F \\
& + [240\psi'(\frac{1}{3}) - 160\pi^2 + 3456\zeta(3) - 14832] C_A T_F N_f \\
& \left. - 864C_F T_F N_f + 648C_F^2 + 1152T_F^2 N_f^2 \right] \frac{C_F a^2}{432} + O(a^4) . \quad (5.31)
\end{aligned}$$

For comparison with $\overline{\text{MS}}$ and the other two MOM schemes we note the full arbitrary linear gauge values are

$$\begin{aligned}
\beta^{\text{MOMh}}(a, \alpha) = & - [11.0000000 - 0.6666667N_f] a^2 \\
& - [102.0000000 - 14.0146029\alpha - 2.7070116\alpha^2 + 1.3710348\alpha^3 \\
& - [12.6666667 - 1.4373952\alpha - 0.6093488\alpha^2] N_f] a^3
\end{aligned}$$

$$\begin{aligned}
& - \left[2813.4929483 - 138.6933593\alpha - 35.8596355\alpha^2 + 4.4867569\alpha^3 \right. \\
& \quad \left. + 9.6466346\alpha^4 + 1.3338552\alpha^5 \right. \\
& \quad \left. - \left[617.6471542 - 40.9610609\alpha - 14.8518881\alpha^2 - 2.2357888\alpha^3 \right. \right. \\
& \quad \quad \left. \left. - 0.2607456\alpha^4 \right] N_f + 21.5028181N_f^2 \right] a^4 + O(a^5) \\
\gamma_A^{\text{MOMh}}(a, \alpha) &= [0.6666667N_f - 6.5000000 + 1.5000000\alpha]a \\
& - \left[34.7278768 - 3.9421899\alpha - 3.4864362\alpha^2 + 1.8105174\alpha^3 \right. \\
& \quad \left. - \left[8.1900386 - 1.4373952\alpha - 0.8046744\alpha^2 \right] N_f \right] a^2 \\
& - \left[1195.1013833 - 28.3678253\alpha - 8.6742911\alpha^2 - 5.5422477\alpha^3 \right. \\
& \quad \left. + 10.3727415\alpha^4 + 1.0931120\alpha^5 \right. \\
& \quad \left. - \left[323.9161114 - 37.7892337\alpha - 11.8481326\alpha^2 - 1.7973717\alpha^3 \right. \right. \\
& \quad \quad \left. \left. - 0.1108276\alpha^4 \right] N_f + [12.2638016 - 0.5937675\alpha] N_f^2 \right] a^3 + O(a^4) \\
\gamma_c^{\text{MOMh}}(a, \alpha) &= [0.7500000\alpha - 2.2500000]a \\
& - \left[9.0788804 + 3.5599978\alpha - 0.5362065\alpha^2 - 0.2197413\alpha^3 \right. \\
& \quad \left. - 0.7500000N_f \right] a^2 \\
& - \left[462.6953814 - 89.8527064\alpha - 1.2245066\alpha^2 + 2.9101869\alpha^3 \right. \\
& \quad \left. - 0.3430542\alpha^4 - 0.1126679\alpha^5 \right. \\
& \quad \left. - \left[76.4163746 - 1.4615295\alpha - 0.0175290\alpha^2 \right] N_f + 2.5000000N_f^2 \right] a^3 \\
& + O(a^4) \\
\gamma_\psi^{\text{MOMh}}(a, \alpha) &= 1.3333333\alpha a \\
& + \left[22.3333333 + 0.0467440\alpha + 2.1252097\alpha^2 + 0.3906512\alpha^3 \right. \\
& \quad \left. - 1.3333333N_f \right] a^2 \\
& + \left[260.0472082 - 162.9606897\alpha - 38.3957984\alpha^2 + 2.9734643\alpha^3 \right. \\
& \quad \left. + 1.2107696\alpha^4 + 0.2002985\alpha^5 \right. \\
& \quad \left. - \left[47.3050219 - 12.9383858\alpha - 2.4062325\alpha^2 \right] N_f + 0.8888889N_f^2 \right] a^3 \\
& + O(a^4) . \tag{5.32}
\end{aligned}$$

The β -function has the same N_f polynomial structure as the $\overline{\text{MS}}$ scheme.

6 Quark-gluon vertex.

Finally, we complete our task with the quark-gluon vertex and the MOMq scheme expressions. The $\overline{\text{MS}}$ amplitudes are

$$\begin{aligned}
\Sigma_{(1)}^{\text{qqv}}(p, q) \Big|_{\overline{\text{MS}}} &= 1 + \left[4.3162206 - 0.5887601\alpha - 0.4570116\alpha^2 \right] a \\
& + \left[89.2876778 - 2.5488660\alpha + 0.7959457\alpha^2 + 0.2344278\alpha^3 \right. \\
& \quad \left. + 0.3427587\alpha^4 - [12.1366772 + 0.9766280\alpha + 0.5077907\alpha^2]N_f \right] a^2 \\
& + O(a^3)
\end{aligned}$$

$$\begin{aligned}
\Sigma_{(2)}^{\text{qqv}}(p, q) \Big|_{\overline{\text{MS}}} &= \Sigma_{(5)}^{\text{qqv}}(p, q) \Big|_{\overline{\text{MS}}} \\
&= \left[2.5980335 - 2.3056953\alpha - 0.4140232\alpha^2 \right] a \\
&\quad + \left[26.4812470 - 21.7488508\alpha - 5.3984938\alpha^2 + 0.4547874\alpha^3 \right. \\
&\quad \left. + 0.3105174\alpha^4 - [6.2718940 + 1.0339459\alpha + 0.4600258\alpha^2]N_f \right] a^2 \\
&\quad + O(a^3) \\
\Sigma_{(3)}^{\text{qqv}}(p, q) \Big|_{\overline{\text{MS}}} &= \Sigma_{(4)}^{\text{qqv}}(p, q) \Big|_{\overline{\text{MS}}} \\
&= \left[2.0502686 - 2.5226306\alpha - 0.5000000\alpha^2 \right] a \\
&\quad + \left[12.7352941 - 25.2299763\alpha - 6.6819786\alpha^2 + 0.0320681\alpha^3 \right. \\
&\quad \left. + 0.3750000\alpha^4 - [4.8715927 + 0.9193101\alpha + 0.5555555\alpha^2]N_f \right] a^2 \\
&\quad + O(a^3) \\
\Sigma_{(6)}^{\text{qqv}}(p, q) \Big|_{\overline{\text{MS}}} &= - \left[4.3622718 + 2.3439072\alpha + 0.5859768\alpha^2 \right] a \\
&\quad - \left[131.9911153 + 45.4675027\alpha + 4.8573516\alpha^2 + 1.2207850\alpha^3 \right. \\
&\quad \left. - 0.4394826\alpha^4 - [10.9228497 + 1.9532560\alpha - 0.6510853\alpha^2]N_f \right] a^2 \\
&\quad + O(a^3)
\end{aligned} \tag{6.33}$$

where the symmetry of the exchange of the two external quark legs is manifest. This emerged from the computation and was not imposed. In Table 2 we have given the comparison table analogous to Table 1 for this vertex. Again the numerical estimates and the results we have produced are virtually identical.

$\Sigma_{(1),2}^{\text{qqv}}(p, q) \Big _{\overline{\text{MS}}}$	C_F^2	$C_F C_A$	C_A^2	$C_A T_F N_f$	$C_F T_F N_f$
Ref [22]	0.206(4)	- 0.20(4)	0.679(1)	- 0.4968(4)	- 0.0211(4)
This paper	0.2048069	- 0.2158263	0.6755204	- 0.4958508	- 0.0221492

Table 2. Comparison of channel 1 two loop Landau gauge amplitude with [22] by colour factor.

The corresponding MOMq scheme expressions are

$$\begin{aligned}
\Sigma_{(1)}^{\text{qqv}}(p, q) \Big|_{\text{MOMq}} &= 1 + O(a^3) \\
\Sigma_{(2)}^{\text{qqv}}(p, q) \Big|_{\text{MOMq}} &= \Sigma_{(5)}^{\text{qqv}}(p, q) \Big|_{\text{MOMq}} \\
&= \left[2.5980335 - 2.3056953\alpha - 0.4140232\alpha^2 \right] a \\
&\quad - \left[28.1605810 - 15.7267125\alpha + 11.9916906\alpha^2 + 5.1627786\alpha^3 \right. \\
&\quad \left. + 0.5676402\alpha^4 + [3.3851901 + 1.0339459\alpha]N_f \right] a^2 + O(a^3) \\
\Sigma_{(3)}^{\text{qqv}}(p, q) \Big|_{\text{MOMq}} &= \Sigma_{(4)}^{\text{qqv}}(p, q) \Big|_{\text{MOMq}} \\
&= \left[2.0502686 - 2.5226306\alpha - 0.5000000\alpha^2 \right] a \\
&\quad - \left[30.3859453 - 13.4480431\alpha + 14.1587135\alpha^2 + 6.3930196\alpha^3 \right. \\
&\quad \left. + 0.6855174\alpha^4 + [2.5935165 + 0.9193101\alpha]N_f \right] a^2 + O(a^3)
\end{aligned}$$

$$\begin{aligned}
\Sigma_{(6)}^{\text{qqv}}(p, q) \Big|_{\text{MOMq}} &= - \left[4.3622718 + 2.3439072\alpha + 0.5859768\alpha^2 \right] a \\
&- \left[40.2438356 + 27.9113517\alpha + 15.1059206\alpha^2 + 7.9109326\alpha^3 \right. \\
&\quad \left. + 0.8033946\alpha^4 - [6.0758810 + 1.9532560\alpha]N_f \right] a^2 \\
&+ O(a^3)
\end{aligned} \tag{6.34}$$

where clearly channel 1 correctly corresponds to the MOMq scheme definition as it is the only channel to contain the divergences in ϵ . The quark external leg interchange also correctly emerges as a minor check on our computations corresponding to swapping p and q . From the renormalization constants in each scheme the numerical conversion functions from MOMq to $\overline{\text{MS}}$ for $SU(3)$ are

$$\begin{aligned}
C_g^{\text{MOMq}}(a, \alpha) &= 1 - \left[8.3578873 - 1.1720934\alpha - 0.0820116\alpha^2 - 0.5555556N_f \right] a \\
&- \left[131.2981279 + 7.8598968\alpha - 0.3923795\alpha^2 + 0.7219823\alpha^3 \right. \\
&\quad \left. + 0.0943409\alpha^4 - [27.6257962 + 1.6277910\alpha + 0.1366860\alpha^2]N_f \right. \\
&\quad \left. + 0.1543230N_f^2 \right] a^2 + O(a^3) \\
C_A^{\text{MOMq}}(a, \alpha) &= 1 + \left[8.0833333 + 1.5000000\alpha + 0.7500000\alpha^2 - 1.1111111N_f \right] a \\
&+ \left[256.1034914 + 31.4182743\alpha + 8.4375000\alpha^2 + 2.8125000\alpha^3 \right] \\
&- [53.9129277 - 1.6666667\alpha]N_f + 1.2345679N_f^2 \Big] a^2 + O(a^3) \\
C_c^{\text{MOMq}}(a, \alpha) &= 1 + 3.0000000a \\
&+ \left[80.9357699 + 3.0725670\alpha + 1.3465290\alpha^2 - 5.9375000N_f \right] a^2 \\
&+ O(a^3) \\
C_\psi^{\text{MOMq}}(a, \alpha) &= 1 - 1.3333333\alpha a \\
&- \left[25.4642061 + 11.5753172\alpha + 2.7222222\alpha^2 - 2.3333333N_f \right] a^2 \\
&+ O(a^3) .
\end{aligned} \tag{6.35}$$

The analytic form of the first produces the coupling constant mapping which is

$$\begin{aligned}
a_{\text{MOMq}} &= a_{\overline{\text{MS}}} \\
&+ \left[\left[52\pi^2 - 78\psi'(\tfrac{1}{3}) + 993 \right] C_A - 240T_F N_f + \left[48\psi'(\tfrac{1}{3}) - 32\pi^2 - 432 \right] C_F \right] \frac{a_{\overline{\text{MS}}}^2}{108} \\
&+ \left[\left[9768\pi^2\psi'(\tfrac{1}{3}) - 7326(\psi'(\tfrac{1}{3}))^2 - 350910\psi'(\tfrac{1}{3}) + 2133\psi'''(\tfrac{1}{3}) \right. \right. \\
&\quad \left. + 664848s_2(\tfrac{\pi}{6}) - 1329696s_2(\tfrac{\pi}{2}) - 1108080s_3(\tfrac{\pi}{6}) + 886464s_3(\tfrac{\pi}{2}) \right. \\
&\quad \left. - 8944\pi^4 + 233940\pi^2 - 54432\Sigma - 164754\zeta(3) + 3278628 \right. \\
&\quad \left. + 4617\frac{\ln^2(3)\pi}{\sqrt{3}} - 55404\frac{\ln(3)\pi}{\sqrt{3}} - 4959\frac{\pi^3}{\sqrt{3}} \right] C_A^2 \\
&+ \left[27360(\psi'(\tfrac{1}{3}))^2 - 36480\pi^2\psi'(\tfrac{1}{3}) - 174816\psi'(\tfrac{1}{3}) + 1296\psi'''(\tfrac{1}{3}) \right. \\
&\quad \left. - 1959552s_2(\tfrac{\pi}{6}) + 3919104s_2(\tfrac{\pi}{2}) + 3265920s_3(\tfrac{\pi}{6}) - 2612736s_3(\tfrac{\pi}{2}) \right. \\
&\quad \left. + 8704\pi^4 + 116544\pi^2 + 46656\zeta(3) - 1960848 \right. \\
&\quad \left. - 13608\frac{\ln^2(3)\pi}{\sqrt{3}} + 163296\frac{\ln(3)\pi}{\sqrt{3}} + 14616\frac{\pi^3}{\sqrt{3}} \right] C_A C_F
\end{aligned}$$

$$\begin{aligned}
& + \left[26112\pi^2\psi'(\tfrac{1}{3}) - 19584(\psi'(\tfrac{1}{3}))^2 + 295488\psi'(\tfrac{1}{3}) - 7776\psi'''(\tfrac{1}{3}) \right. \\
& \quad - 373248s_2(\tfrac{\pi}{6}) + 746496s_2(\tfrac{\pi}{2}) + 622080s_3(\tfrac{\pi}{6}) - 497664s_3(\tfrac{\pi}{2}) \\
& \quad + 12032\pi^4 - 196992\pi^2 + 31104\Sigma + 435456\zeta(3) + 256608 \\
& \quad \left. - 2592\frac{\ln^2(3)\pi}{\sqrt{3}} + 31104\frac{\ln(3)\pi}{\sqrt{3}} + 2784\frac{\pi^3}{\sqrt{3}} \right] C_F^2 \\
& + \left[197568\psi'(\tfrac{1}{3}) - 864\psi'''(\tfrac{1}{3}) + 186624s_2(\tfrac{\pi}{6}) - 373248s_2(\tfrac{\pi}{2}) - 311040s_3(\tfrac{\pi}{6}) \right. \\
& \quad + 248832s_3(\tfrac{\pi}{2}) + 2304\pi^4 - 131712\pi^2 - 217728\zeta(3) - 1549440 \\
& \quad \left. + 1296\frac{\ln^2(3)\pi}{\sqrt{3}} - 15552\frac{\ln(3)\pi}{\sqrt{3}} - 1392\frac{\pi^3}{\sqrt{3}} \right] C_A T_F N_f + 115200 T_F^2 N_f^2 \\
& + \left[44544\pi^2 - 66816\psi'(\tfrac{1}{3}) + 373248\zeta(3) + 80352 \right] C_F T_F N_f \frac{a_{\overline{\text{MS}}}^3}{23328} \\
& + O\left(a_{\overline{\text{MS}}}^4\right)
\end{aligned} \tag{6.36}$$

in the Landau gauge. Numerically we have

$$\begin{aligned}
a_{\text{MOMq}} &= a_{\overline{\text{MS}}} + \left[16.7157746 - 2.3441868\alpha_{\overline{\text{MS}}} - 0.1640232\alpha_{\overline{\text{MS}}}^2 - 1.1111111N_f \right] a_{\overline{\text{MS}}}^2 \\
&+ \left[472.1590953 - 43.0575532\alpha_{\overline{\text{MS}}} - 0.7760123\alpha_{\overline{\text{MS}}}^2 + 2.0207162\alpha_{\overline{\text{MS}}}^3 \right. \\
&\quad + 0.2088596\alpha_{\overline{\text{MS}}}^4 - \left[83.1112168 - 0.6513959\alpha_{\overline{\text{MS}}} \right] N_f \\
&\quad \left. + 1.2345679N_f^2 \right] a_{\overline{\text{MS}}}^3 + O\left(a_{\overline{\text{MS}}}^4\right).
\end{aligned} \tag{6.37}$$

Thus, similar to the previous sections the explicit three loop Landau gauge renormalization group functions are

$$\begin{aligned}
\beta^{\text{MOMq}}(a, 0) &= - \left[\frac{11}{3}C_A - \frac{4}{3}T_F N_f \right] a^2 - \left[\frac{34}{3}C_A^2 - 4C_F T_F N_f - \frac{20}{3}C_A T_F N_f \right] a^3 \\
&+ \left[\left[71478(\psi'(\tfrac{1}{3}))^2 - 95304\pi^2\psi'(\tfrac{1}{3}) - 40266\psi'(\tfrac{1}{3}) - 7821\psi'''(\tfrac{1}{3}) \right. \right. \\
&\quad - 2437776s_2(\tfrac{\pi}{6}) + 4875552s_2(\tfrac{\pi}{2}) + 4062960s_3(\tfrac{\pi}{6}) - 3250368s_3(\tfrac{\pi}{2}) \\
&\quad + 52624\pi^4 + 26844\pi^2 + 199584\Sigma + 604098\zeta(3) - 3593970 \\
&\quad \left. - 16929\frac{\ln^2(3)\pi}{\sqrt{3}} + 203148\frac{\ln(3)\pi}{\sqrt{3}} + 18183\frac{\pi^3}{\sqrt{3}} \right] C_A^3 \\
&+ \left[206976\pi^2\psi'(\tfrac{1}{3}) - 155232(\psi'(\tfrac{1}{3}))^2 + 1951776\psi'(\tfrac{1}{3}) - 4752\psi'''(\tfrac{1}{3}) \right. \\
&\quad + 7185024s_2(\tfrac{\pi}{6}) - 14370048s_2(\tfrac{\pi}{2}) - 11975040s_3(\tfrac{\pi}{6}) + 9580032s_3(\tfrac{\pi}{2}) \\
&\quad - 56320\pi^4 - 1301184\pi^2 - 171072\zeta(3) - 159408 \\
&\quad \left. + 49896\frac{\ln^2(3)\pi}{\sqrt{3}} - 598752\frac{\ln(3)\pi}{\sqrt{3}} - 53592\frac{\pi^3}{\sqrt{3}} \right] C_A^2 C_F \\
&+ \left[34656\pi^2\psi'(\tfrac{1}{3}) - 25992(\psi'(\tfrac{1}{3}))^2 - 392328\psi'(\tfrac{1}{3}) + 6012\psi'''(\tfrac{1}{3}) \right. \\
&\quad + 202176s_2(\tfrac{\pi}{6}) - 404352s_2(\tfrac{\pi}{2}) - 336960s_3(\tfrac{\pi}{6}) + 269568s_3(\tfrac{\pi}{2}) \\
&\quad - 27584\pi^4 + 261552\pi^2 - 72576\Sigma + 578664\zeta(3) + 3133080 \\
&\quad \left. + 1404\frac{\ln^2(3)\pi}{\sqrt{3}} - 16848\frac{\ln(3)\pi}{\sqrt{3}} - 1508\frac{\pi^3}{\sqrt{3}} \right] C_A^2 T_F N_f \\
&+ \left[88704(\psi'(\tfrac{1}{3}))^2 - 118272\pi^2\psi'(\tfrac{1}{3}) - 1387584\psi'(\tfrac{1}{3}) + 28512\psi'''(\tfrac{1}{3}) \right. \\
&\quad \left. + 1368576s_2(\tfrac{\pi}{6}) - 2737152s_2(\tfrac{\pi}{2}) - 2280960s_3(\tfrac{\pi}{6}) + 1824768s_3(\tfrac{\pi}{2}) \right]
\end{aligned}$$

$$\begin{aligned}
& - 36608\pi^4 + 925056\pi^2 - 114048\Sigma - 1596672\zeta(3) + 427680 \\
& + 9504\frac{\ln^2(3)\pi}{\sqrt{3}} - 114048\frac{\ln(3)\pi}{\sqrt{3}} - 10208\frac{\pi^3}{\sqrt{3}} \Big] C_A C_F^2 \\
& + \Big[56448(\psi'(\tfrac{1}{3}))^2 - 75264\pi^2\psi'(\tfrac{1}{3}) - 592704\psi'(\tfrac{1}{3}) + 1728\psi'''(\tfrac{1}{3}) \\
& - 2612736s_2(\tfrac{\pi}{6}) + 5225472s_2(\tfrac{\pi}{2}) + 4354560s_3(\tfrac{\pi}{6}) - 3483648s_3(\tfrac{\pi}{2}) \\
& + 20480\pi^4 + 395136\pi^2 - 1306368\zeta(3) + 1194912 \\
& - 18144\frac{\ln^2(3)\pi}{\sqrt{3}} + 217728\frac{\ln(3)\pi}{\sqrt{3}} + 19488\frac{\pi^3}{\sqrt{3}} \Big] C_A C_F T_F N_f \\
& + \Big[163584\psi'(\tfrac{1}{3}) - 1152\psi'''(\tfrac{1}{3}) + 248832s_2(\tfrac{\pi}{6}) - 497664s_2(\tfrac{\pi}{2}) \\
& - 414720s_3(\tfrac{\pi}{6}) + 331776s_3(\tfrac{\pi}{2}) + 3072\pi^4 - 109056\pi^2 \\
& - 290304\zeta(3) - 585792 + 1728\frac{\ln^2(3)\pi}{\sqrt{3}} \\
& - 20736\frac{\ln(3)\pi}{\sqrt{3}} - 1856\frac{\pi^3}{\sqrt{3}} \Big] C_A T_F^2 N_f^2 \\
& + \Big[43008\pi^2\psi'(\tfrac{1}{3}) - 32256(\psi'(\tfrac{1}{3}))^2 + 463104\psi'(\tfrac{1}{3}) - 10368\psi'''(\tfrac{1}{3}) \\
& - 497664s_2(\tfrac{\pi}{6}) + 995328s_2(\tfrac{\pi}{2}) + 829440s_3(\tfrac{\pi}{6}) - 663552s_3(\tfrac{\pi}{2}) \\
& + 13312\pi^4 - 308736\pi^2 + 41472\Sigma + 580608\zeta(3) + 171072 \\
& - 3456\frac{\ln^2(3)\pi}{\sqrt{3}} + 41472\frac{\ln(3)\pi}{\sqrt{3}} + 3712\frac{\pi^3}{\sqrt{3}} \Big] C_F^2 T_F N_f \\
& + \Big[18432\pi^2 - 27648\psi'(\tfrac{1}{3}) + 497664\zeta(3) - 352512 \Big] C_F T_F^2 N_f^2 N_f \Big] \frac{a^4}{23328} \\
& + O(a^5)
\end{aligned} \tag{6.38}$$

$$\begin{aligned}
\gamma_A^{\text{MOMq}}(a, 0) = & - [13C_A - 8T_F N_f] \frac{a}{6} \\
& + \Big[[338\pi^2 - 507\psi'(\tfrac{1}{3}) + 864] C_A^2 + [312\psi'(\tfrac{1}{3}) - 208\pi^2 - 108] C_A T_F N_f \\
& + [312\psi'(\tfrac{1}{3}) - 208\pi^2 - 2808] C_A C_F \\
& + [128\pi^2 - 192\psi'(\tfrac{1}{3}) + 3024] C_F T_F N_f \Big] \frac{a^2}{324} \\
& + \Big[[548808\pi^2\psi'(\tfrac{1}{3}) - 411606(\psi'(\tfrac{1}{3}))^2 + 4914\psi'(\tfrac{1}{3}) + 27729\psi'''(\tfrac{1}{3}) \\
& + 8643024s_2(\tfrac{\pi}{6}) - 17286048s_2(\tfrac{\pi}{2}) - 14405040s_3(\tfrac{\pi}{6}) + 11524032s_3(\tfrac{\pi}{2}) \\
& - 256880\pi^4 - 3276\pi^2 - 707616\Sigma + 1016226\zeta(3) - 2542266 \\
& + 60021\frac{\ln^2(3)\pi}{\sqrt{3}} - 720252\frac{\ln(3)\pi}{\sqrt{3}} - 64467\frac{\pi^3}{\sqrt{3}} \Big] C_A^3 \\
& + \Big[745056(\psi'(\tfrac{1}{3}))^2 - 993408\pi^2\psi'(\tfrac{1}{3}) - 8587296\psi'(\tfrac{1}{3}) + 16848\psi'''(\tfrac{1}{3}) \\
& - 25474176s_2(\tfrac{\pi}{6}) + 50948352s_2(\tfrac{\pi}{2}) + 42456960s_3(\tfrac{\pi}{6}) \\
& - 33965568s_3(\tfrac{\pi}{2}) + 286208\pi^4 + 5724864\pi^2 + 606528\zeta(3) - 198288 \\
& - 176904\frac{\ln^2(3)\pi}{\sqrt{3}} + 2122848\frac{\ln(3)\pi}{\sqrt{3}} + 190008\frac{\pi^3}{\sqrt{3}} \Big] C_A^2 C_F \\
& + \Big[253296(\psi'(\tfrac{1}{3}))^2 - 337728\pi^2\psi'(\tfrac{1}{3}) + 1856304\psi'(\tfrac{1}{3}) - 28296\psi'''(\tfrac{1}{3}) \\
& - 2892672s_2(\tfrac{\pi}{6}) + 5785344s_2(\tfrac{\pi}{2}) + 4821120s_3(\tfrac{\pi}{6}) - 3856896s_3(\tfrac{\pi}{2})
\end{aligned}$$

$$\begin{aligned}
& + 188032\pi^4 - 1237536\pi^2 + 435456\Sigma + 3059856\zeta(3) + 1166400 \\
& - 20088\frac{\ln^2(3)\pi}{\sqrt{3}} + 241056\frac{\ln(3)\pi}{\sqrt{3}} + 21576\frac{\pi^3}{\sqrt{3}} \Big] C_A^2 T_F N_f \\
& + \Big[499200\pi^2\psi'(\tfrac{1}{3}) - 374400(\psi'(\tfrac{1}{3}))^2 + 5997888\psi'(\tfrac{1}{3}) - 101088\psi'''(\tfrac{1}{3}) \\
& - 4852224s_2(\tfrac{\pi}{6}) + 9704448s_2(\tfrac{\pi}{2}) + 8087040s_3(\tfrac{\pi}{6}) - 6469632s_3(\tfrac{\pi}{2}) \\
& + 103168\pi^4 - 3998592\pi^2 + 404352\Sigma + 5660928\zeta(3) - 6368544 \\
& - 33696\frac{\ln^2(3)\pi}{\sqrt{3}} + 404352\frac{\ln(3)\pi}{\sqrt{3}} + 36192\frac{\pi^3}{\sqrt{3}} \Big] C_A C_F^2 \\
& + \Big[611328\pi^2\psi'(\tfrac{1}{3}) - 458496(\psi'(\tfrac{1}{3}))^2 + 5660928\psi'(\tfrac{1}{3}) + 10368\psi'''(\tfrac{1}{3}) \\
& + 15676416s_2(\tfrac{\pi}{6}) - 31352832s_2(\tfrac{\pi}{2}) - 26127360s_3(\tfrac{\pi}{6}) \\
& + 20901888s_3(\tfrac{\pi}{2}) - 176128\pi^4 - 3773952\pi^2 \\
& - 8584704\zeta(3) + 7309440 + 108864\frac{\ln^2(3)\pi}{\sqrt{3}} \\
& - 1306368\frac{\ln(3)\pi}{\sqrt{3}} - 116928\frac{\pi^3}{\sqrt{3}} \Big] C_A C_F T_F N_f \\
& + \Big[6912\psi'''(\tfrac{1}{3}) - 981504\psi'(\tfrac{1}{3}) - 1492992s_2(\tfrac{\pi}{6}) + 2985984s_2(\tfrac{\pi}{2}) \\
& + 2488320s_3(\tfrac{\pi}{6}) - 1990656s_3(\tfrac{\pi}{2}) - 18432\pi^4 + 654336\pi^2 \\
& - 1244160\zeta(3) + 155520 + 10368\frac{\ln^2(3)\pi}{\sqrt{3}} \\
& + 124416\frac{\ln(3)\pi}{\sqrt{3}} + 11136\frac{\pi^3}{\sqrt{3}} \Big] C_A T_F^2 N_f^2 \\
& + \Big[230400\pi^2\psi'(\tfrac{1}{3}) - 307200(\psi'(\tfrac{1}{3}))^2 - 4188672\psi'(\tfrac{1}{3}) + 62208\psi'''(\tfrac{1}{3}) \\
& + 2985984s_2(\tfrac{\pi}{6}) - 597198s_2(\tfrac{\pi}{2}) - 4976640s_3(\tfrac{\pi}{6}) + 3981312s_3(\tfrac{\pi}{2}) \\
& - 63488\pi^4 + 2792448\pi^2 - 248832\Sigma - 3483648\zeta(3) + 8118144 \\
& + 20736\frac{\ln^2(3)\pi}{\sqrt{3}} - 248832\frac{\ln(3)\pi}{\sqrt{3}} - 22272\frac{\pi^3}{\sqrt{3}} \Big] C_F^2 T_F N_f \\
& + \Big[165888\psi'(\tfrac{1}{3}) - 110592\pi^2 + 2985984\zeta(3) \\
& - 3608064 \Big] C_F T_F^2 N_f^2 \Big] \frac{a^3}{139968} + O(a^4) \tag{6.39}
\end{aligned}$$

$$\begin{aligned}
\gamma_c^{\text{MOMq}}(a, 0) &= -\frac{3}{4}C_A a \\
&+ \Big[\Big[26\pi^2 - 39\psi'(\tfrac{1}{3}) + 90 \Big] C_A + 36T_F N_f \\
&+ \Big[24\psi'(\tfrac{1}{3}) - 16\pi^2 - 216 \Big] C_F \Big] \frac{C_A a^2}{72} \\
&+ \Big[\Big[42216\pi^2\psi'(\tfrac{1}{3}) - 31662(\psi'(\tfrac{1}{3}))^2 + 15066\psi'(\tfrac{1}{3}) + 2133\psi'''(\tfrac{1}{3}) \\
&+ 664848s_2(\tfrac{\pi}{6}) - 1329696s_2(\tfrac{\pi}{2}) - 1108080s_3(\tfrac{\pi}{6}) + 886464s_3(\tfrac{\pi}{2}) \\
&- 19760\pi^4 - 10044\pi^2 - 54432\Sigma + 40338\zeta(3) - 267786 \\
&+ 4617\frac{\ln^2(3)\pi}{\sqrt{3}} - 55404\frac{\ln(3)\pi}{\sqrt{3}} - 4959\frac{\pi^3}{\sqrt{3}} \Big] C_A^2 \\
&+ \Big[57312(\psi'(\tfrac{1}{3}))^2 - 76416\pi^2\psi'(\tfrac{1}{3}) - 669600\psi'(\tfrac{1}{3}) + 1296\psi'''(\tfrac{1}{3})
\end{aligned}$$

$$\begin{aligned}
& - 1959552s_2(\tfrac{\pi}{6}) + 3919104s_2(\tfrac{\pi}{2}) + 3265920s_3(\tfrac{\pi}{6}) \\
& - 2612736s_3(\tfrac{\pi}{2}) + 22016\pi^4 + 446400\pi^2 + 46656\zeta(3) + 66096 \\
& - 13608\frac{\ln^2(3)\pi}{\sqrt{3}} + 163296\frac{\ln(3)\pi}{\sqrt{3}} + 14616\frac{\pi^3}{\sqrt{3}} \Big] C_A C_F \\
& + \left[38400(\psi'(\tfrac{1}{3}))^2 - 28800\pi^2\psi'(\tfrac{1}{3}) + 461376\psi'(\tfrac{1}{3}) - 7776\psi'''(\tfrac{1}{3}) \right. \\
& - 373248s_2(\tfrac{\pi}{6}) + 746496s_2(\tfrac{\pi}{2}) + 622080s_3(\tfrac{\pi}{6}) - 497664s_3(\tfrac{\pi}{2}) \\
& + 7936\pi^4 - 307584\pi^2 + 31104\Sigma + 435456\zeta(3) - 489888 \\
& - 2592\frac{\ln^2(3)\pi}{\sqrt{3}} + 31104\frac{\ln(3)\pi}{\sqrt{3}} + 2784\frac{\pi^3}{\sqrt{3}} \Big] C_F^2 - 103680T_F^2 N_f^2 \\
& + \left[145152\psi'(\tfrac{1}{3}) - 864\psi'''(\tfrac{1}{3}) + 186624s_2(\tfrac{\pi}{6}) - 373248s_2(\tfrac{\pi}{2}) \right. \\
& - 311040s_3(\tfrac{\pi}{6}) + 248832s_3(\tfrac{\pi}{2}) + 2304\pi^4 - 96768\pi^2 - 15552\zeta(3) \\
& + 273456 + 1296\frac{\ln^2(3)\pi}{\sqrt{3}} - 15552\frac{\ln(3)\pi}{\sqrt{3}} - 1392\frac{\pi^3}{\sqrt{3}} \Big] C_A T_F N_f \\
& + \left[23040\pi^2 - 34560\psi'(\tfrac{1}{3}) + 264384 \right] C_F T_F N_f \Big] \frac{C_A a^3}{31104} + O(a^4) \quad (6.40)
\end{aligned}$$

and

$$\begin{aligned}
\gamma_\psi^{\text{MOMq}}(a, 0) &= [25C_A - 6C_F - 8T_F N_f] \frac{C_F a^2}{4} \\
&+ \left[\left[3900\psi'(\tfrac{1}{3}) - 2600\pi^2 - 13230\zeta(3) + 10287 \right] C_A^2 \right. \\
&+ \left[2224\pi^2 - 3336\psi'(\tfrac{1}{3}) + 5184\zeta(3) + 16092 \right] C_A C_F \\
&+ \left[832\pi^2 - 1248\psi'(\tfrac{1}{3}) + 3456\zeta(3) - 8784 \right] C_A T_F N_f \\
&+ \left[768\psi'(\tfrac{1}{3}) - 512\pi^2 - 7776 \right] C_F T_F N_f + 1152T_F^2 N_f^2 \\
&+ \left[576\psi'(\tfrac{1}{3}) - 384\pi^2 - 4536 \right] C_F^2 \Big] \frac{C_F a^2}{432} + O(a^4) . \quad (6.41)
\end{aligned}$$

The corresponding arbitrary linear covariant $SU(3)$ expressions are

$$\begin{aligned}
\beta^{\text{MOMq}}(a, \alpha) &= - [11.0000000 - 0.6666667N_f] a^2 \\
&- \left[102.0000000 + 15.2372141\alpha - 1.3839787\alpha^2 - 0.4920696\alpha^3 \right. \\
&- \left. \left[12.6666667 + 1.5627912\alpha + 0.2186976\alpha^2 \right] N_f \right] a^3 \\
&- \left[1843.6527285 + 422.0731852\alpha + 123.3734958\alpha^2 - 19.5130255\alpha^3 \right. \\
&- 3.5055190\alpha^4 - 0.0961314\alpha^5 \\
&- \left. \left[588.6548455 + 60.5454812\alpha + 16.3955703\alpha^2 + 0.9282359\alpha^3 \right. \right. \\
&- \left. \left. 0.0000059\alpha^4 \right] N_f + 22.5878118N_f^2 \right] a^4 + O(a^5) \\
\gamma_A^{\text{MOMq}}(a, \alpha) &= [0.6666667N_f - 6.5000000 + 1.5000000\alpha] a \\
&- \left[46.6391320 + 22.5608759\alpha - 6.2001294\alpha^2 + 0.8789652\alpha^3 \right. \\
&- \left. \left[9.4117058 + 1.5627912\alpha - 0.3906512\alpha^2 \right] N_f \right] a^2 \\
&- \left[2027.7437143 + 333.3082218\alpha + 184.2382915\alpha^2 - 24.3519716\alpha^3 \right. \\
&- \left. 12.6718858\alpha^4 + 1.9200786\alpha^5 \right]
\end{aligned}$$

$$\begin{aligned}
& - \left[415.6990152 + 49.4053081\alpha + 9.7003844\alpha^2 - 3.7908552\alpha^3 \right. \\
& \quad \left. - 0.4783683\alpha^4 \right] N_f \\
& + [11.1788079 - 1.3021707\alpha] N_f^2 \Big] a^3 + O(a^4) \\
\gamma_c^{\text{MOMq}}(a, \alpha) &= [0.7500000\alpha - 2.2500000]a \\
& - \left[13.2020072 + 12.3112512\alpha - 2.5140879\alpha^2 - 0.6855174\alpha^3 \right. \\
& \quad \left. - 0.7500000N_f \right] a^2 \\
& - \left[740.1341645 + 1.8665778\alpha + 100.6450352\alpha^2 - 3.4355918\alpha^3 \right. \\
& \quad \left. - 8.7678000\alpha^4 - 1.0965129\alpha^5 \right. \\
& \quad \left. - \left[75.5032720 + 4.1186469\alpha + 1.7109768\alpha^2 \right] N_f + 2.5000000N_f^2 \right] a^3 \\
& + O(a^4) \\
\gamma_\psi^{\text{MOMq}}(a, \alpha) &= 1.3333333\alpha a \\
& + \left[22.3333333 + 2.4900784\alpha + 8.1255824\alpha^2 + 1.2186976\alpha^3 \right. \\
& \quad \left. - 1.3333333N_f \right] a^2 \\
& + \left[341.8989103 + 182.9132891\alpha + 43.9801061\alpha^2 + 74.9283461\alpha^3 \right. \\
& \quad \left. + 21.4352687\alpha^4 + 1.9493562\alpha^5 \right. \\
& \quad \left. - \left[52.1916907 - 3.1076278\alpha - 2.1669462\alpha^2 \right] N_f + 0.8888889N_f^2 \right] a^3 \\
& + O(a^4) . \tag{6.42}
\end{aligned}$$

Unlike the previous two sections we close this section with a few remarks on other renormalization group functions in QCD. Recently, there has been interest in the RI'/SMOM renormalization scheme which was developed in [49]. Briefly the scheme was introduced to renormalize flavour non-singlet quark currents in a quark 2-point function at the symmetric subtraction point. The motivation for such a scheme is that it circumvents potential infrared singularities of the measurement of the same Green's function on the lattice in the chiral limit for exceptional momentum configurations, [20]. More specifically the scalar, vector and tensor currents have been considered at one and two loops in this scheme, [49, 43, 50], and the associated conversion functions for each operator relative to the $\overline{\text{MS}}$ scheme have been determined. Thus the RI'/SMOM three loop operator anomalous dimensions have been deduced in an arbitrary covariant gauge. More recently, the full set of amplitudes has been provided for each of these currents at two loops in [51]. Equally the same information has been provided for the $n = 2$ and 3 moments of the flavour non-singlet Wilson operator central to deep inelastic scattering, [52, 53]. As an extension of the RI'/SMOM analysis we have determined the three loop anomalous dimensions of the scalar and tensor quark currents as well as the $n = 2$ moment of the Wilson operator in the MOMq scheme. Numerically the $SU(3)$ values are

$$\begin{aligned}
\gamma_S^{\text{MOMq}}(a, \alpha) &= -4.0000000a - \left[7.5709424 + 8.3450254\alpha + 0.3121855\alpha^2 + 1.7918763N_f \right] a^2 \\
& + \left[324.9490278 + 84.8639648\alpha - 37.0694357\alpha^2 - 1.1713670\alpha^3 \right. \\
& \quad \left. + 0.2647488\alpha^4 - \left[16.2843867 + 18.7759399\alpha + 0.5878184\alpha^2 \right] N_f \right. \\
& \quad \left. - 2.6666667N_f^2 \right] a^3 + O(a^4) \tag{6.43}
\end{aligned}$$

$$\gamma_T^{\text{MOMq}}(a, \alpha) = 1.3333333a + \left[20.3014252 + 8.1573041\alpha + 1.8959382\alpha^2 - 0.5878931N_f \right] a^2$$

$$\begin{aligned}
& + \left[276.1622997 + 149.4337760\alpha + 83.3978821\alpha^2 + 32.6152122\alpha^3 \right. \\
& \quad \left. + 3.1873834\alpha^4 - \left[61.3516342 + 4.4008934\alpha + 0.1928562\alpha^2 \right] N_f \right. \\
& \quad \left. + 2.0740741N_f^2 \right] a^3 + O(a^4)
\end{aligned} \tag{6.44}$$

and

$$\begin{aligned}
\gamma_{W_2,11}^{\text{MOMq}}(a, \alpha) &= 3.5555556a + \left[45.2685593 + 15.7105153\alpha + 3.0417366\alpha^2 - 2.6264345N_f \right] a^2 \\
&+ \left[818.7944256 + 154.0549414\alpha + 59.1199636\alpha^2 + 6.2844353\alpha^3 \right. \\
&\quad \left. + 0.3234930\alpha^4 - \left[140.9726462 + 5.3687957\alpha + 0.8615923\alpha^2 \right] N_f \right. \\
&\quad \left. + 4.7392379N_f^2 \right] a^3 + O(a^4) \\
\gamma_{W_2,12}^{\text{MOMq}}(a, \alpha) &= -1.7777778a - \left[16.5859286 + 7.5010250\alpha + 0.9582634\alpha^2 - 0.7715739N_f \right] a^2 \\
&- \left[298.1271156 + 48.6705419\alpha + 27.5758422\alpha^2 + 3.1422177\alpha^3 \right. \\
&\quad \left. + 0.1617465\alpha^4 - \left[54.9281680 + 0.1449714\alpha + 0.2531120\alpha^2 \right] N_f \right. \\
&\quad \left. + 2.1752211N_f^2 \right] a^3 + O(a^4) \\
\gamma_{W_2,22}^{\text{MOMq}}(a, \alpha) &= O(a^4)
\end{aligned} \tag{6.45}$$

where S and T respectively denote scalar and tensor currents. The Wilson operator second moment three loop anomalous dimension corresponds to the 11 entry of the dimension two upper triangular mixing matrix, [52]. The 22 entry corresponds to the total derivative operator for this operator moment but its anomalous dimension is equivalent to that of the vector current, [52]. As the vector current is conserved and hence a physical operator its anomalous dimension is zero to all orders in perturbation theory consistent with the Slavnov-Taylor identity. Whilst we have included a subset of the operators considered in recent RI'/SMOM analyses for comparison we have not recorded the complete set for the MOMq or either of the other MOM schemes discussed here. The reason for this is simple. In deriving (6.43), (6.44) and (6.45) we have arrived at the results by constructing the respective conversion functions and then using the analogous relation to (3.9) for operator renormalization. It transpires, like our observation for the wave function and gauge parameter conversion functions, that the respective conversion functions are formally the same as those already given in [49, 43, 50] for the quark currents and that for W_2 given in [52]. The only difference here in producing (6.43), (6.44) and (6.45) is that the coupling constant mapping between the schemes is not trivial as it is in the RI'/SMOM case. Therefore, if one is interested in the structure of the three loop anomalous dimensions in the MOMggg and MOMh schemes it is merely a simple exercise to construct them from the known conversion functions. We should add that in deriving the two loop parts of (6.43), (6.44) and (6.45) we have not only used the renormalization group construction but also carried out the explicit two loop renormalization of each of the three operators in the MOMq scheme. Therefore, the two loop terms of each have been derived directly and also indirectly from the respective one loop conversion functions which we have constructed explicitly.

7 Discussion.

We close with some remarks and give an overall perspective on our computations. First, we have computed the two loop structure of each of the trivalent QCD vertices with a linear covariant gauge fixing at the symmetric subtraction point for the $\overline{\text{MS}}$ scheme as well as the three MOM

schemes. Although we have only given the MOMggg, MOMh and MOMq versions for each of the respective triple gluon, ghost-gluon and quark-gluon vertices, the values of say the triple gluon vertex in the MOMq scheme, if of interest, can be readily deduced from the results given here via the Slavnov-Taylor identities of QCD, [21]. In addition we have deduced the basic three loop renormalization group functions including the β -functions in each of the three MOM schemes in an arbitrary linear covariant gauge. As there has been numerical estimates for some of these quantities in the Landau gauge in [22] we need to comment on how our results compare in addition to earlier remarks. First, we consider the $SU(3)$ Landau gauge two loop mapping of the coupling constant from the MOMi schemes to $\overline{\text{MS}}$. Specifically, the polynomials in N_f of the perturbative map were given in [22] and the coefficients were denoted by d_{lj}^{MOMi} , where l is the loop order and j is the power of N_f in the polynomial. As d_{10}^{MOMi} , d_{11}^{MOMi} and d_{22}^{MOMi} can all be computed analytically without approximation and which we agree with for MOMggg and MOMq, we focus on the remaining two, d_{20}^{MOMi} and d_{21}^{MOMi} , which could only be evaluated numerically in [22]. In Table 3 we give the results of [22] in its notation, in order to preserve the error estimates, together with the numerical evaluation of our analytic results for $SU(3)$ in the same convention. Aside from the MOMh scheme the agreement is remarkably close. However, that for d_{21}^{MOMh} differs by about 5% whilst that for d_{20}^{MOMh} is significantly different. We believe that this is a transcription inconsistency in the presentation of the results in [22] akin to that noted earlier here and in [28] for the amplitudes of this vertex. This is also because we do not tally with the one loop coefficients d_{10}^{MOMh} and d_{11}^{MOMh} which are known exactly, [21]. We have checked that we obtain the *same* one loop MOMh ghost-gluon vertex counterterm given in [21] *exactly* for all α . Equally we have obtained the same vertex counterterm as [21] for the quark-gluon vertex as a check on our overall procedures which were the same for each vertex. For instance, the one loop master integral denoted by I in [21] and [22] arises in d_{10}^{MOMh} . It derives directly from the vertex counterterm only and the coefficient of I is therefore in a one-to-one mapping to the coefficient of I in d_{10}^{MOMh} . Comparing the counterterms of our results and [22], it is clear that the coefficients differ but we find our expressions are consistent with [21]. Therefore, the one loop agreement of our MOMh results with [21] suggests that we have a more credible MOMh coupling constant mapping as we have followed the same symbolic algebraic algorithm as that for MOMggg and MOMq.

	d_{20}^{MOMggg}	d_{21}^{MOMggg}	d_{20}^{MOMh}	d_{21}^{MOMh}	d_{20}^{MOMq}	d_{21}^{MOMq}
Ref [22]	59.8(8)	− 12.6(2)	35.88(2)	− 5.0707(6)	29.53(1)	− 5.1961(4)
This paper	60.02892	− 12.63031	40.12125	− 5.34725	29.50994	− 5.19445

Table 3. Comparison of two of the two loop $SU(3)$ coupling constant relation coefficients for each scheme in the notation and convention of [22].

Concerning the final Landau gauge $SU(3)$ β -functions of [22], we have already briefly noted the accuracy of our exact expressions in [24]. Summarizing, for the coefficient of the N_f independent three loop term the agreement was around 2% for the MOMggg and MOMh β -functions and less than 0.2% for MOMq. In Table 4 we have presented the coefficients of the polynomial in N_f of the three loop term of the β -function in each scheme. These are denoted by β_{lj} where again l is the loop order and j is the power of N_f in the polynomial. Only the MOMggg scheme has a cubic term in N_f and it is known analytically. So the precise numerical agreement merely reflects this. The linear term in N_f in the MOMggg scheme is virtually zero because of an accidental cancellation for the colour group $SU(3)$ which was noted in [22]. This is the reason why a large error was recorded for this coefficient though we obtain the same sign for our coefficient. Overall for MOMggg and MOMq all the coefficients are very close to the central

values given in [22]. That for $\beta_{30}^{\text{MOMggg}}$ had a relatively large error, possibly as a result of the purely gluonic graphs which might have a more slow convergence in the asymmetric expansion parameter approach used in [22]. However, it is still reassuring that our result emerges so close to the central value in this particular case which roughly has a 2% error. Therefore, in this context the slightly smaller error on β_{30}^{MOMh} suggests that ultimately the MOMh β -function of [22] has been correctly obtained. Although the agreement of the other N_f polynomial coefficients is reasonably decent there does not appear to be as close an overlap compared to say those for MOMq. This is probably, because, as we noted earlier, the amplitude for the triple gluon and quark-gluon vertices for the channel containing the poles in ϵ , or equivalently the channel corresponding to the vertex Feynman rule, are in very good agreement with the Landau gauge estimates given in [22]. Therefore, overall we are confident that our results are correct.

Scheme	Article	β_{30}	β_{31}	β_{32}	β_{33}
MOMggg	Ref [22]	24(2)	0.04(63)	− 1.05(3)	0.0415330
	This paper	24.5466309	0.0088426	− 1.0482740	0.0415330
MOMh	Ref [22]	44.82(5)	− 9.730(5)	0.3276(1)	-
	This paper	43.9608273	− 9.6507368	0.3359815	-
MOMq	Ref [22]	28.86(3)	− 9.206(3)	0.35322(7)	-
	This paper	28.8070739	− 9.1977320	0.3529346	-

Table 4. Comparison of coefficients of three loop Landau gauge β -function in the convention of [22].

One final point worth emphasising in this regard is to note that when the computations of [22] were being carried out the technology available in terms of algorithms, such as [40], and computer power was not comparable to that of current levels. Therefore it is a remarkable achievement that overall the results are in solid agreement. This is partly because the expansion technique of [22] to evaluate a 3-point Green’s function at the symmetric point in terms of 2-point functions to allow the application of MINCER, requires a large amount of integration by parts especially as the powers of the propagators increases in each term of the expansion. Despite this, however, it would seem that the method of [22] could be extended in principle to tackle the construction of the vertex functions to three loops. Whilst there are now three loop anomalous dimensions and β -functions available as intermediate checks on any numerical estimates, it would provide a reasonable estimate for *four* loop renormalization group functions including the β -functions in physical MOM schemes. Such higher order computations may be necessary soon in order to not only assist with lattice matching of vertex functions but also for more accurate determination of the value of the strong coupling constant.

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A Tensor basis and projectors.

In this appendix we list in turn the basis of tensors used as well as the coefficients for the projection of each of the three vertices. Away from the symmetric subtraction point the tensor basis is enlarged.

A.1 Triple gluon vertex.

For the triple gluon vertex we have the basis tensors

$$\begin{aligned}
\mathcal{P}_{(1)\mu\nu\sigma}^{\text{ggg}}(p, q) &= \eta_{\mu\nu}p_\sigma, \quad \mathcal{P}_{(2)\mu\nu\sigma}^{\text{ggg}}(p, q) = \eta_{\nu\sigma}p_\mu, \quad \mathcal{P}_{(3)\mu\nu\sigma}^{\text{ggg}}(p, q) = \eta_{\sigma\mu}p_\nu \\
\mathcal{P}_{(4)\mu\nu\sigma}^{\text{ggg}}(p, q) &= \eta_{\mu\nu}q_\sigma, \quad \mathcal{P}_{(5)\mu\nu\sigma}^{\text{ggg}}(p, q) = \eta_{\nu\sigma}q_\mu, \quad \mathcal{P}_{(6)\mu\nu\sigma}^{\text{ggg}}(p, q) = \eta_{\sigma\mu}q_\nu \\
\mathcal{P}_{(7)\mu\nu\sigma}^{\text{ggg}}(p, q) &= \frac{1}{\mu^2}p_\mu p_\nu p_\sigma, \quad \mathcal{P}_{(8)\mu\nu\sigma}^{\text{ggg}}(p, q) = \frac{1}{\mu^2}p_\mu p_\nu q_\sigma, \quad \mathcal{P}_{(9)\mu\nu\sigma}^{\text{ggg}}(p, q) = \frac{1}{\mu^2}p_\mu q_\nu p_\sigma \\
\mathcal{P}_{(10)\mu\nu\sigma}^{\text{ggg}}(p, q) &= \frac{1}{\mu^2}q_\mu p_\nu p_\sigma, \quad \mathcal{P}_{(11)\mu\nu\sigma}^{\text{ggg}}(p, q) = \frac{1}{\mu^2}p_\mu q_\nu q_\sigma, \quad \mathcal{P}_{(12)\mu\nu\sigma}^{\text{ggg}}(p, q) = \frac{1}{\mu^2}q_\mu p_\nu q_\sigma \\
\mathcal{P}_{(13)\mu\nu\sigma}^{\text{ggg}}(p, q) &= \frac{1}{\mu^2}q_\mu q_\nu p_\sigma, \quad \mathcal{P}_{(14)\mu\nu\sigma}^{\text{ggg}}(p, q) = \frac{1}{\mu^2}q_\mu q_\nu q_\sigma.
\end{aligned} \tag{A.1}$$

The first six tensors represent the structures which appear in the triple gluon vertex Feynman rule. As there are fourteen tensors for this vertex then in order to simplify the presentation we have partitioned the projection matrix into a 3×3 block matrix with partitions of sizes 6, 4 and 4. Defining

$$\mathcal{M}^{\text{ggg}} = -\frac{1}{27(d-2)} \begin{pmatrix} \mathcal{M}_{11}^{\text{ggg}} & \mathcal{M}_{12}^{\text{ggg}} & \mathcal{M}_{13}^{\text{ggg}} \\ \mathcal{M}_{21}^{\text{ggg}} & \mathcal{M}_{22}^{\text{ggg}} & \mathcal{M}_{23}^{\text{ggg}} \\ \mathcal{M}_{31}^{\text{ggg}} & \mathcal{M}_{32}^{\text{ggg}} & \mathcal{M}_{33}^{\text{ggg}} \end{pmatrix}$$

then in d -dimensions the sub-matrices for the projection are

$$\begin{aligned}
\mathcal{M}_{11}^{\text{ggg}} &= \begin{pmatrix} 36 & 0 & 0 & 18 & 0 & 0 \\ 0 & 36 & 0 & 0 & 18 & 0 \\ 0 & 0 & 36 & 0 & 0 & 18 \\ 18 & 0 & 0 & 36 & 0 & 0 \\ 0 & 18 & 0 & 0 & 36 & 0 \\ 0 & 0 & 18 & 0 & 0 & 36 \end{pmatrix}, \quad \mathcal{M}_{12}^{\text{ggg}} = \begin{pmatrix} 48 & 24 & 24 & 24 \\ 48 & 24 & 24 & 24 \\ 48 & 24 & 24 & 24 \\ 24 & 48 & 12 & 12 \\ 24 & 12 & 12 & 48 \\ 24 & 12 & 48 & 12 \end{pmatrix} \\
\mathcal{M}_{13}^{\text{ggg}} &= \begin{pmatrix} 12 & 12 & 48 & 24 \\ 48 & 12 & 12 & 24 \\ 12 & 48 & 12 & 24 \\ 24 & 24 & 24 & 48 \\ 24 & 24 & 24 & 48 \\ 24 & 24 & 24 & 48 \end{pmatrix}, \quad \mathcal{M}_{21}^{\text{ggg}} = \begin{pmatrix} 48 & 48 & 48 & 24 & 24 & 24 \\ 24 & 24 & 24 & 48 & 12 & 12 \\ 24 & 24 & 24 & 12 & 12 & 48 \\ 24 & 24 & 24 & 12 & 48 & 12 \end{pmatrix} \\
\mathcal{M}_{22}^{\text{ggg}} &= \begin{pmatrix} 64(d+1) & 32(d+1) & 32(d+1) & 32(d+1) \\ 32(d+1) & 32(2d-1) & 16(d+1) & 16(d+1) \\ 32(d+1) & 16(d+1) & 32(2d-1) & 16(d+1) \\ 32(d+1) & 16(d+1) & 16(d+1) & 32(2d-1) \end{pmatrix} \\
\mathcal{M}_{23}^{\text{ggg}} &= \begin{pmatrix} 16(d+4) & 16(d+4) & 16(d+4) & 8(d+10) \\ 8(4d+1) & 8(4d+1) & 8(d+4) & 16(d+4) \\ 8(4d+1) & 8(d+4) & 8(4d+1) & 16(d+4) \\ 8(d+4) & 8(4d+1) & 8(4d+1) & 16(d+4) \end{pmatrix} \\
\mathcal{M}_{31}^{\text{ggg}} &= \begin{pmatrix} 12 & 48 & 12 & 24 & 24 & 24 \\ 12 & 12 & 48 & 24 & 24 & 24 \\ 48 & 12 & 12 & 24 & 24 & 24 \\ 24 & 24 & 24 & 48 & 48 & 48 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}_{32}^{\text{ggg}} &= \begin{pmatrix} 16(d+4) & 8(4d+1) & 8(4d+1) & 8(d+4) \\ 16(d+4) & 8(4d+1) & 8(d+4) & 8(4d+1) \\ 16(d+4) & 8(d+4) & 8(4d+1) & 8(4d+1) \\ 8(d+10) & 16(d+4) & 16(d+4) & 16(d+4) \end{pmatrix} \\
\mathcal{M}_{33}^{\text{ggg}} &= \begin{pmatrix} 32(2d-1) & 16(d+1) & 16(d+1) & 32(d+1) \\ 16(d+1) & 32(2d-1) & 16(d+1) & 32(d+1) \\ 16(d+1) & 16(d+1) & 32(2d-1) & 32(d+1) \\ 32(d+1) & 32(d+1) & 32(d+1) & 64(d+1) \end{pmatrix}
\end{aligned} \tag{A.2}$$

in d -dimensions.

A.2 Ghost-gluon vertex.

At the symmetric subtraction point there are two basis tensors for the ghost-gluon vertex which are

$$\mathcal{P}_{(1)\sigma}^{\text{cgg}}(p, q) = p_\sigma \quad , \quad \mathcal{P}_{(2)\sigma}^{\text{cgg}}(p, q) = q_\sigma . \tag{A.3}$$

Hence, in this case the projection matrix is relatively simple and is given by

$$\mathcal{M}^{\text{cgg}} = -\frac{1}{3} \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} \tag{A.4}$$

in d -dimensions.

A.3 Quark-gluon vertex.

The quark-gluon vertex tensor basis involves six independent tensors. These are the same as those used for the vector current insertion in a quark 2-point function in [51] and are given by

$$\begin{aligned}
\mathcal{P}_{(1)\sigma}^{\text{qqg}}(p, q) &= \gamma_\sigma \quad , \quad \mathcal{P}_{(2)\sigma}^{\text{qqg}}(p, q) = \frac{p_\sigma \not{p}}{\mu^2} \quad , \quad \mathcal{P}_{(3)\sigma}^{\text{qqg}}(p, q) = \frac{p_\sigma \not{q}}{\mu^2} \quad , \\
\mathcal{P}_{(4)\sigma}^{\text{qqg}}(p, q) &= \frac{q_\sigma \not{p}}{\mu^2} \quad , \quad \mathcal{P}_{(5)\sigma}^{\text{qqg}}(p, q) = \frac{q_\sigma \not{q}}{\mu^2} \quad , \quad \mathcal{P}_{(6)\sigma}^{\text{qqg}}(p, q) = \frac{1}{\mu^2} \Gamma_{(3)\sigma pq}
\end{aligned} \tag{A.5}$$

where the generalized d -dimensional γ -matrices denoted by $\Gamma_{(n)}^{\mu_1 \dots \mu_n}$ were defined earlier. Equally the same projection tensor in d -dimensions emerges as in [51] which is

$$\mathcal{M}^{\text{qqg}} = \frac{1}{36(d-2)} \begin{pmatrix} 9 & 12 & 6 & 6 & 12 & 0 \\ 12 & 16(d-1) & 8(d-1) & 8(d-1) & 4(d+2) & 0 \\ 6 & 8(d-1) & 4(4d-7) & 4(d-1) & 8(d-1) & 0 \\ 6 & 8(d-1) & 4(d-1) & 4(4d-7) & 8(d-1) & 0 \\ 12 & 4(d+2) & 8(d-1) & 8(d-1) & 16(d-1) & 0 \\ 0 & 0 & 0 & 0 & 0 & -12 \end{pmatrix} . \tag{A.6}$$

The partition of the projection matrix into the $\Gamma_{(1)}^\mu$ and $\Gamma_{(3)}^{\mu\nu\sigma}$ sectors is apparent.

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